

Integrable Wilson loops

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Abstract

The generalized quark–antiquark potential of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory on $\mathbb{S}^3 \times \mathbb{R}$ calculates the potential between a pair of heavy charged particles separated by an arbitrary angle on \mathbb{S}^3 and also an angle in flavor space. It can be calculated by a Wilson loop following a prescribed path and couplings, or after a conformal transformation, by a cusped Wilson loop in flat space, hence also generalizing the usual concept of the cusp anomalous dimension. In $AdS_5 \times \mathbb{S}^5$ this is calculated by an infinite open string. I present here an open spin–chain model which calculates the spectrum of excitations of such open strings. In the dual gauge theory these are cusped Wilson loops with extra operator insertions at the cusp. The boundaries of the spin–chain introduce a non-trivial reflection phase and break the bulk symmetry down to a single copy of $\mathfrak{psu}(2|2)$. The dependence on the two angles is captured by the two embeddings of this algebra into $\mathfrak{psu}(2|2)^2$, *i.e.*, by a global rotation. The exact answer to this problem is conjectured to be given by solutions to a set of twisted boundary thermodynamic Bethe ansatz integral equations. In particular the generalized quark–antiquark potential or cusp anomalous dimension is recovered by calculating the ground state energy of the minimal length spin–chain, with no sites. It gets contributions only from virtual particles reflecting off the boundaries. I reproduce from this calculation some known weak coupling perturbative results.

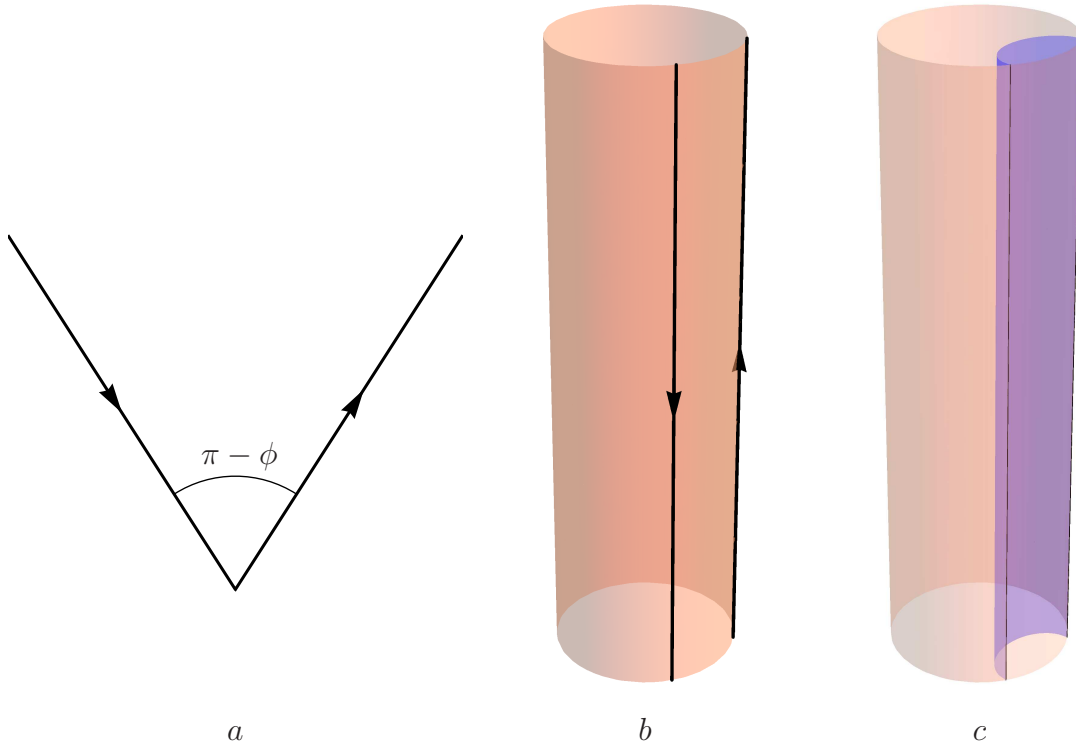


Figure 1: A cusped Wilson loop (a) in $\mathcal{N} = 4$ SYM on \mathbb{R}^4 (and by analytic continuation also on $\mathbb{R}^{3,1}$) is related to a pair of antiparallel lines (b) on $S^3 \times \mathbb{R}$. In the dual string theory this is calculated by a string world-sheet in $AdS_5 \times S^5$ ending along the two lines on the boundary (c).

1 Introduction

Wilson loops are some of the most important observables in nonabelian gauge theories. Among their many features, they capture the potential between heavy probe particles, hence can serve as an order parameter for confinement [1]. They calculate a big part of the effect of high-energy scattering of charged particles [2, 3], and in the case of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) are conjectured to calculate scattering amplitudes exactly [4, 5].

In [6] a family of Wilson loop operators in $\mathcal{N} = 4$ SYM were presented and studied. Two useful points of view on these operators are (see Figure 1)

- A cusp of angle $\pi - \phi$ in \mathbb{R}^4 , *i.e.*, two rays meeting at a point.
- Two lines along the time direction on $S^3 \times \mathbb{R}$, separated by an angle $\pi - \phi$ on the sphere.

These two configurations are related to each other by a conformal transformation, so are essentially equivalent. The Maldacena-Wilson loops of $\mathcal{N} = 4$ SYM also include a coupling to a real scalar field and it is natural to allow the two rays/lines to couple to two different

scalars, say Φ^1 and $\Phi^1 \cos \theta + \Phi^2 \sin \theta$. Thus giving a two parameter family of observables, with the opening angle $\pi - \phi$ (such that $\phi = 0$ is the straight line) and θ .

These geometries calculate the physical quantities mentioned above. The pair of antiparallel lines gives the potential $V(\lambda, \phi, \theta)$ between two heavy charged particles propagating over a large time T on \mathbb{S}^3

$$\langle W_{\text{lines}} \rangle = e^{-TV(\lambda, \phi, \theta)}. \quad (1.1)$$

It is possible to recover the potential in flat space by taking residue of V as $\phi \rightarrow \pi$ (see [6, 7] for a detailed discussion). Cusped Wilson loops have the structure

$$\langle W_{\text{cusp}} \rangle = e^{-\Gamma_{\text{cusp}}(\lambda, \phi, \theta) \log \frac{\Lambda}{\epsilon}}, \quad (1.2)$$

with the same function $\Gamma_{\text{cusp}}(\lambda, \phi, \theta) = V(\lambda, \phi, \theta)$ and Λ and ϵ are IR and UV cutoffs, respectively. Scattering amplitudes are related to cusped Wilson loops in Minkowski space, so one should take $\phi = i\varphi$ to be imaginary. In particular in the limit of $\varphi \rightarrow \infty$, leads to γ_{cusp} — the *universal cusp anomalous dimension*

$$\Gamma_{\text{cusp}}(\lambda, i\varphi, \theta) \rightarrow \frac{\varphi \gamma_{\text{cusp}}(\lambda)}{4}. \quad (1.3)$$

As explained in detail in [8], in the limit of small ϕ and vanishing θ , $\Gamma_{\text{cusp}}(\lambda, \phi, 0) \sim -B(\lambda)\phi^2$ calculates also the radiation of a particle moving along an arbitrary smooth path. An exact expression is given there for B including $1/N$ corrections to all orders in the gauge coupling λ . A modification of this expression applies also in expanding Γ_{cusp} around the BPS configurations $\phi = \pm\theta$ [9]. In this manuscript an integrable spin-chain model is presented which is conjectured to calculate the full function Γ_{cusp} in the planar approximation.

There is another application of this quantity, where it was originally defined [10, 11]. Wilson loops satisfy a set of equations known as the loop equations [12]. These equations are nontrivial only for intersecting loops, where cusps appear. The loop equations have been solved in zero dimensional matrix models and in two dimensional Yang-Mills. To define them properly in four dimensional theories requires understanding cusped Wilson loops and how to renormalize their divergences. Logarithmic divergences of Wilson loops (and in the case of the Maldacena-Wilson loops in $\mathcal{N} = 4$ SYM, all the divergences) come from cusps, or from insertions of local operators. As explained in the following, integrability supplies the answer to this question in $\mathcal{N} = 4$ SYM, where all the logarithmic divergences arising from either insertions or cusps are governed by the same spin-chain model (to be precise, a set of twisted boundary thermodynamic Bethe ansatz equations (TBTBA, BTBA, or TBA in the following)).

Integrability in this calculation is most easily seen in the string theory dual, where at leading order the Wilson loop is calculated by a classical string solution [13, 14, 15, 16, 6] and of course the string sigma-model in $AdS_5 \times S^5$ is integrable [17]. At the one-loop level all the fluctuation operators about these particular classical solutions turn out to be

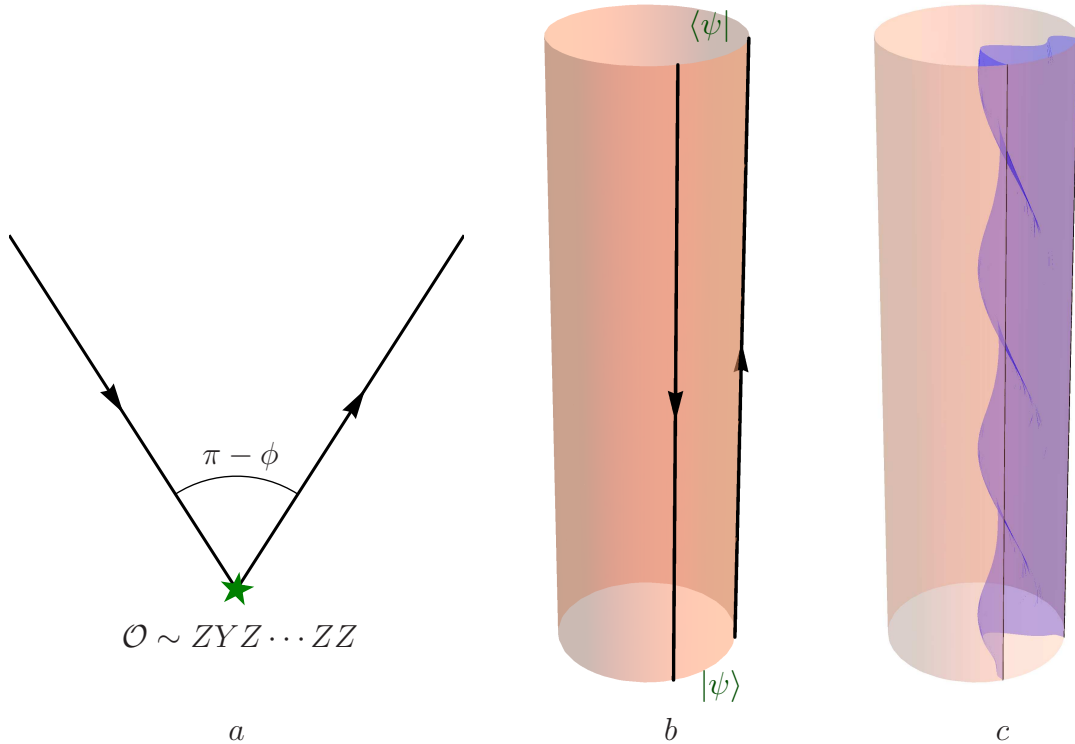


Figure 2: A generalization of Figure 1 allows an arbitrary adjoint valued local operator to be inserted at the apex of the cusp (a) in \mathbb{R}^4 . After the conformal transformation this is a pair of antiparallel lines (b) on $S^3 \times \mathbb{R}$, seemingly like in Figure 1b, but in fact the details of the operator \mathcal{O} are represented by nontrivial boundary condition $|\psi\rangle$ at past and future infinity. The dual string solution in $AdS_5 \times S^5$ still ends along the two lines, but is in an excited state (in the figure the sphere is suppressed).

versions of a known integrable system, the Lamé differential operator [18, 19, 20, 6]. The calculations of the same quantity on the gauge theory side were based on brute force Feynman diagrammatics.

In order to see the integrability of these operators also in the gauge theory, it proves useful to consider a further generalization of the set of observables. Along a Wilson loop in the fundamental representation one can insert local operators in the adjoint representation of the gauge group while retaining gauge invariance. We shall insert such an operator at the apex of the cusp (Figure 2). In the $S^3 \times \mathbb{R}$ picture this operator is at past infinity, and corresponds to an excitation of the Wilson loop.¹

¹The infinite lines are not well defined observables, as they are not invariant under gauge transformations at infinity. One should supply extra boundary conditions to construct them, and usually one assumes that

As mentioned, the expectation value of cusped Wilson loops suffer from logarithmic divergences, which can be considered the conformal dimension of the operator, meaning that these observable are eigenstates of the dilation operator. The generalized quark-antiquark potential V in (1.1) (and Γ_{cusp} in (1.2)) is exactly that conformal dimension. In order to understand this dimension as the solution of a spin-chain problem it is useful to consider the more general situation.

Note that in order to make this argument it is crucial that geometrically the cusp is invariant under dilations and the antiparallel lines under translations (also in addition a $U(1)$ of rotations in the transverse space and an $SO(4)$ flavor symmetry). Therefore one can define the problem of finding operators in the quantum theory which are eigenstates under this transformation. A similar argument cannot be implemented in a trivial way for Wilson loops of other shapes.

Another way to visualize the situation in the dual string theory, where the Wilson loops are described by a string stretching between the two lines at the boundary of $AdS_5 \times S^5$. Thus, along the boundary of the open string we have two different boundary conditions, corresponding to the two lines. In world-sheet language we should consider the string with the insertion of two operators, these are not regular open string states, rather they are *boundary changing operators*. To be precise, for any pair of boundary conditions there is a family of such boundary changing operators exactly corresponding to the cusped Wilson loops with extra insertions. The original cusp without an extra operator insertion maps to the ground state of the open string, *i.e.*, the boundary changing operator of the lowest dimension.

The case of the insertion of adjoint operators into the straight Wilson line with $\theta = \phi = 0$ was studied in [21]. In the string theory dual these operators are mapped to regular open-string vertex operators as the boundaries of the string do not get modified. On the gauge theory side the dimension of these insertions can be calculated by solving an open spin-chain problem, as is reviewed in the next section and extend to the more general situation.

If the operators carry macroscopic amount of angular momentum in AdS_5 or S^5 , then there is a semiclassical description for them in string theory. For the insertions into the straight line this was done in [21], and it can be generalized for an arbitrary cusp using the techniques in [22]. These solutions are such that the string approaches the boundary along the same two lines as the solutions without the insertions (see Appendix B of [6]), but performing extra rotations around the S^5 or in AdS_5 in the bulk of the world-sheet.

To be completely explicit, I write down the form of the Wilson loop operator with inser-

the lines close onto each other in a smooth way. The excitations considered here have alternative boundary conditions.

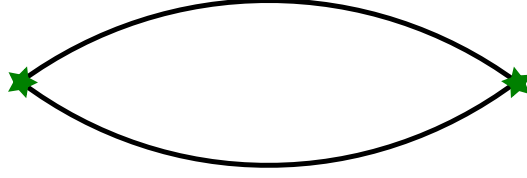


Figure 3: Yet another picture of a Wilson loop with two cusps (with possible local insertions) connected by arcs. It is related to that in Figure 2a by a conformal transformation mapping the point at infinity to finite distance. If the distance between the cusps is d , the expectation value of this Wilson loop is $\langle W \rangle \propto 1/d^{2V(\lambda, \phi, \theta)}$, where the logarithmic divergences in (1.2) are interpreted, as usual as renormalizing the classical dimension.

tion of the simplest adjoint valued local operator $\mathcal{O} = Z^J$ with $Z = \Phi^5 + i\Phi^6$

$$\begin{aligned}
W &= \mathcal{P} e^{\int_{-\infty}^0 [iA_\mu(x(s))\dot{x}^\mu + \Phi^1(x(s))|\dot{x}|] ds} Z(0)^J e^{\int_0^\infty (iA_\mu(x(s))\dot{x}^\mu + \cos \theta \Phi^1(x(s))|\dot{x}| + \sin \theta \Phi^2(x(s))|\dot{x}|) ds} \\
&= \mathcal{P} e^{\int_{-\infty}^0 [iA_1(x(s)) + \Phi^1(x(s))] ds} Z(0)^J e^{\int_0^\infty (i \cos \phi A_1(x(s)) + i \sin \phi A_2(x(s)) + \cos \theta \Phi^1(x(s)) + \sin \theta \Phi^2(x(s))) ds} \\
x(s) &= \begin{cases} (s, 0, 0, 0) & s \leq 0, \\ (s \cos \phi, s \sin \phi, 0, 0) & s \geq 0, \end{cases} \tag{1.4}
\end{aligned}$$

By a conformal transformation the point at infinity can be mapped to finite distance (see Figure 3) and then the Wilson loop is a completely kosher gauge invariant operator with two cusps with local operators Z^J inserted at one and \bar{Z}^J at the other.

$V(\lambda, \phi, \theta)$ was calculated in [6] to second order in perturbation theory.² The result is

$$\begin{aligned}
V(\lambda, \phi, \theta) &= -\frac{\lambda}{8\pi^2} \frac{\cos \theta - \cos \phi}{\sin \phi} \phi \\
&\quad + \left(\frac{\lambda}{8\pi^2} \right)^2 \left[\frac{1}{3} \frac{\cos \theta - \cos \phi}{\sin \phi} (\pi^2 - \phi^2) \phi \right. \\
&\quad \left. - \frac{(\cos \theta - \cos \phi)^2}{\sin^2 \phi} \left(\text{Li}_3(e^{2i\phi}) - \zeta(3) - i\phi \left(\text{Li}_2(e^{2i\phi}) + \frac{\pi^2}{6} \right) + \frac{i}{3}\phi^3 \right) \right], \tag{1.5}
\end{aligned}$$

In the next section the spin-chain model is presented. Section 3 calculates the leading weak coupling contribution to the the cusped Wilson loop from the exchange of a single virtual magnon (Lüscher-Bajnok-Janik correction). Section 4 presents the twisted boundary TBA which calculates this quantity at all values of the coupling. Finally Section 5 discusses some of the results and possible generalizations. Some technical details are relegated to appendices.

At an advanced state of this project I learnt of [23], which has a great deal of overlap with the results reported in this manuscript.

²The three loop answer was derived very recently in [7].

2 Wilson loops and open spin-chains

In this section the relation between the insertion of adjoint valued operators into a Wilson loop and open spin-chains is developed. First the main principles are explained in general terms and only later are the precise formulas for the open spin-chain presented.

In the case of gauge invariant local operators the spin-chain arises [24] by choosing a reference ground-state $\text{Tr } Z^J$ with $Z = \Phi^5 + i\Phi^6$ (also known as the BMN vacuum [25]) and considering the replacement of some of the Z fields by other fields of the theory: The other scalars, the fermions or field-strengths. In addition one can act with derivative operators. The possible insertions are labeled by representations of $PSU(2|2)^2 \subset PSU(2, 2|4)$ which is the residual symmetry preserved by the vacuum. These states are viewed as excitation of a spin-chain and it is conjectured that this spin-chain is integrable and satisfies a particular dispersion relation leading to a solution in principle of the spectral problem of the theory, which has passed many stringent tests. The symmetry is reviewed in Appendix A and the scattering matrix of the fundamental representation of this spin-chain is presented in Appendix B.

The same thing can be done with insertions of adjoint valued local operators into Wilson loops, as explained some time ago in [21]. That paper studied only the insertions of operators made of two complex scalar fields (Z and $Y = \Phi^4 + i\Phi^5$ forming the $SU(2)$ sector) into a straight (or circular) Wilson loop. Here the construction is generalized to the full set of allowed states. Also, the crucial modification of the spin-chain model when replacing a straight line with an arbitrary cusp is explained.

The basic principle of [21], illustrated in Figure 4, is that the calculation of the Wilson loop with local insertions is very similar to that of the dimension of local operators. At one loop order in the planar approximation every field in the local insertion interacts directly only with its two nearest neighbors. Only the first and last fields interact with the Wilson loop itself. This leads naturally to an open spin-chain model, where the bulk hamiltonian is the same as the usual one as in the closed spin-chain, but with two boundaries where one must specify boundary conditions, *i.e.*, reflection matrices.

It is important to emphasize that this is based mainly on abstract arguments and on very few explicit calculations, which is also the case (but to a lesser extent) with the usual spectral problem. It is clear from diagrammatics that indeed the bulk spin-chain hamiltonian is identical to the closed spin-chain one. The hamiltonian in any case is known explicitly only at low orders in perturbation theory, but the scattering matrix is known and therefore expected to be the same. The boundary interactions were calculated in [21] only in the $SU(2)$ sector and only at one loop, where they are essentially trivial. Below, an exact expression for the reflection matrix is proposed, which should capture the boundary interactions at all loop order. It is clearly conjectural, based on the assumption of integrability, representation theory, boundary crossing symmetry and minimality of the dressing factor (see below). The

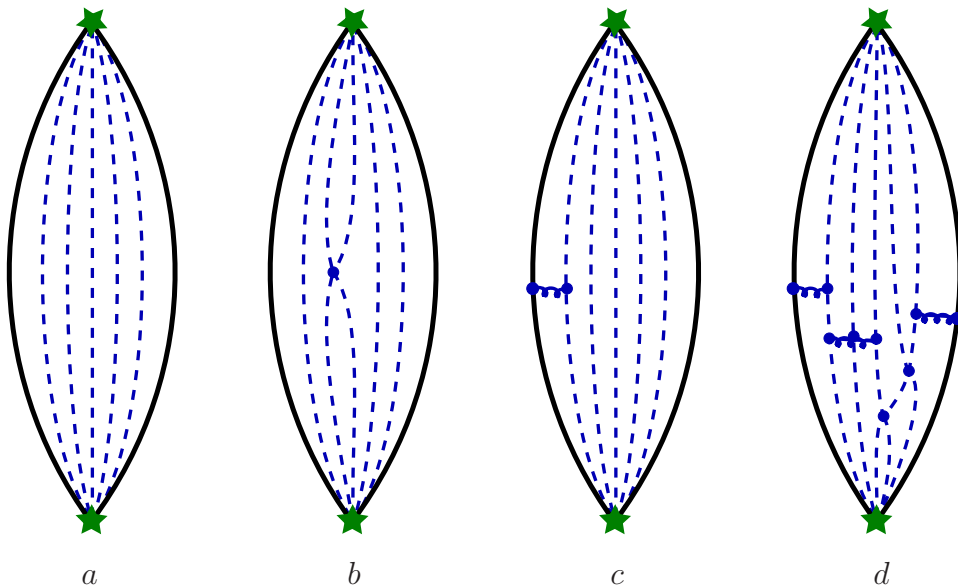


Figure 4: Sample planar Feynman graph calculations for a Wilson loop with two insertion of operators made of five scalar fields at the cusps. At leading order there are only five free propagators (dashed lines) so the classical conformal dimension is five (a). At one loop there are nearest neighbor interactions among the scalar fields (like b), which are identical to those of single trace local operators, but also interaction between the last scalar and the Wilson loop (c). At this order planar graphs connecting the loop to itself are restricted to each of the arcs, they are finite and do not modify the conformal dimension. Finally The two arcs interact by “wrapping effects”, which arise in this example only at six loop order (d). Up to this order the dimension of an operator made of the scalars Z and Y does not depend on the cusp angles ϕ and θ .

formalism presented does reproduce the results of the perturbative calculation, which lends credence to this conjecture.

There are other manifestations of open spin-chains in $\mathcal{N} = 4$ SYM, which apart from the Wilson loops is whenever there are D-branes in the dual $AdS_5 \times S^5$ space, on which the open strings can end.³

The most studied example is that of the D3-brane giant graviton, which is a determinant operator in the field theory, see [26, 27, 28]. A closely related system is that of AdS_5 filling D7-branes introducing fundamental matter into the theory which can serve as the ends of an adjoint valued word, again leading to open spin-chains.

Another system, that of D5-branes representing 3d defects in the gauge theory (domain

³The strings describing the Wilson loop are also open, but extend all the way to the boundary of space, rather than ending on a D-brane.

walls with fundamental matter) [29] has a symmetry much closer to that of the Wilson loops and has been erroneously thought for a long time not to lead to integrable boundary conditions on the spin-chain [30]. This has been corrected recently in [31, 32], and is very useful for the derivation below.

Before going into the technical details, it is useful to explore some of the features that can be extracted from looking at the Feynman diagrams, as in Figure 4.

The modification from the straight line to the cusped loop (or from the circle to the pair of cusps as in Figure 3) is quite easy. The spin-chain has to satisfy two different boundary conditions at the two ends. The two rays of the Wilson loop couple to linear combinations of the scalars Φ^1 and Φ^2 break the $SO(6)$ R -symmetry to $SO(4)$. The ground state Z^J breaks it further to $U(1)$ rotating Φ^3 into Φ^4 . It also has charge J under the $U(1)$ which rotates Φ^5 into Φ^6 .

The fields Z and Y (also \bar{Z} and \bar{Y}) interact with the two segments of the Wilson loop in the same way, at the one loop order they have purely reflective boundary conditions, so the eigenstates of the one loop hamiltonian of the form $\sum_k e^{ipk} Z^{k-1} Y Z^{J-k}$ are Neumann functions. The fields Φ^1 and Φ^2 (or X and \bar{X}) interact differently with the two boundaries, a linear combination of them (depending on the scalar coupling of the Wilson loop) has to vanish at the boundary, so they satisfy partially Dirichlet and partially Neumann conditions and the allowed magnon momenta will depend on θ .

Going to higher order in perturbation theory doesn't change much until order $J + 1$. The range of interaction in the spin-chain is identical to the order in perturbation theory and at this order there are graphs which communicate all the way from one side to the other (Figure 4d). In analogy with the closed spin-chain these can be called "wrapping corrections". For operators made of Z and Y these are the first graphs where the answer will depend on the cusp angles ϕ and θ . In particular the ground state Z^J has dimension $\Delta = J$ up to order λ^J and gets corrections *only* from wrapping effects (and only for $\phi \neq \theta$, when the system is not globally BPS [9]). The wrapping corrections are hard to calculate directly for large J , but in the spin-chain formalism they are given by Lüscher like corrections [33, 34], as discussed in Section 3. Higher order wrapping effects are best captured by the twisted boundary TBA equations.

Note though, that in the case of primary interest, that without the local insertion, $J = 0$, so wrapping corrections contribute at one loop order, double wrapping at two loops and so on. In that case the diagrammatics are not that hard and indeed these calculations were done up to three loop order in [15, 35, 6, 7] (see also [36, 37]). This system then provides an interesting laboratory to study the twisted boundary TBA equations, where the desired quantity is the ground state energy, which is the simplest observable in the TBA. This is discussed in detail in Section 4 below.

2.1 Boundary symmetries

It is now time to start with the detailed analysis of the system. The first question is how the symmetry of the theory and of the usual spin-chain model is modified by the presence of the boundaries. It is crucial that one can focus on one boundary at a time, since the symmetry preserved by each boundary is significantly larger and more restrictive than that preserved by both together.

The $\mathfrak{psu}(2, 2|4)$ supersymmetry algebra of the theory is written down in Appendix A as is its breaking to $\mathfrak{psu}(2|2)^2 \times \mathfrak{u}(1)$ by the choice of ground state. Here we describe the breaking of the symmetry by the Wilson loop, which furnishes the spin-chain with boundaries.

A straight Wilson loop in the x^μ direction and coupling to the scalar Φ^I preserves half the supercharges, those given by the combinations

$$Q_A^\alpha + \epsilon^{\alpha\beta} \gamma_{\beta\dot{\gamma}}^\mu \rho_{AB}^I \bar{Q}^{\dot{\gamma}B}, \quad \bar{S}_{\dot{\alpha}A} - \epsilon_{\dot{\alpha}\dot{\beta}} \gamma^{\mu\dot{\beta}\gamma} \rho_{AB}^I S_\gamma^B. \quad (2.1)$$

Here γ^μ are the usual gamma matrices in space-time and ρ_I are those for the $SO(6)$ flavor symmetry.

These generators close onto an $\mathfrak{osp}(4^*|4)$ algebra. It is possible to include an extra phase $e^{i\alpha}$ between the two terms in both sums and the algebra still closes to an isomorphic algebra. For $\alpha = \pi/2$ this is the symmetry of an 't Hooft loop and other values correspond to dyonic loop operators. So this is the symmetry preserved by each of the boundaries.

Consider the Wilson loop with $\mu = I = 1$ and the choice of $SO(6)$ gamma matrices

$$\begin{aligned} \rho_1^{14} = \rho_1^{23} = 1, & \quad -\rho_3^{13} = \rho_3^{24} = 1, & \quad \rho_5^{34} = \rho_5^{12} = 1, \\ \rho_2^{14} = -\rho_2^{23} = i, & \quad \rho_4^{13} = \rho_4^{24} = i, & \quad \rho_6^{34} = -\rho_6^{12} = i. \end{aligned} \quad (2.2)$$

Of the two copies of $\mathfrak{psu}(2|2)$ annihilating the vacuum Z^J (A.3), only one diagonal copy survives, once considering the linear combinations (2.1) annihilating the Wilson loop. Those are

$$Q_a^\alpha + i(\sigma^3)^\alpha_{\dot{\beta}} (\sigma^3)^{\dot{b}}_a \bar{Q}_{\dot{b}}^{\dot{\beta}}, \quad \bar{S}_{\dot{\alpha}}^{\dot{a}} - i(\sigma^3)^{\beta}_{\dot{\alpha}} (\sigma^3)^{\dot{a}}_b S_\gamma^b. \quad (2.3)$$

A similar algebra, though with a different real form and embedding into $\mathfrak{psu}(2, 2|4)$, namely $\mathfrak{osp}(4|4, \mathbb{R})$, is also preserved by defects represented in $AdS_5 \times \mathbb{S}^5$ by D5-branes. As is shown in [31] the open spin-chain with such boundary conditions has the same boundary symmetry which is the intersection $\mathfrak{osp}(4|4, \mathbb{R}) \cap \mathfrak{psu}(2|2)^2 = \mathfrak{psu}(2|2)$.

In that case the defect breaks the $SO(6)$ R symmetry to $SO(3) \times SO(3)$, while for the Wilson loop it is broken to $SO(5)$. In that case (as in other realizations of open spin-chains in $\mathcal{N} = 4$ SYM) there are two choices for the vacuum, depending on which copy of the two inequivalent $SO(3)$ is broken by the choice of ground state. In one case the resulting boundary is charged under the unbroken $SO(3)$, and therefore carries a representation of $\mathfrak{psu}(2|2)$. In the other case, it is uncharged and does not carry a boundary degree of freedom. In the case of the Wilson loop it is natural to break the $SO(5)$ symmetry to $SO(3)$ and the

resulting boundary has no degrees of freedom. This can be seen by explicit calculation, since the straight Wilson loop with an insertion of Z^J is a perfectly good BPS operator and does not require other fields “shielding” the local operator from the Wilson loop (as would be the case with a vacuum like X^J with $X = \Phi^1 + i\Phi^2$).⁴ This can also be seen from the dual string point of view, where the description of the string ground state [21] does not require breaking the symmetry and the resulting Goldstone bosons of global rotation in $AdS_5 \times S^5$. The spin-chain calculation below can be quite easily generalized to the case of boundary degrees of freedom to describe both type of D5-brane open spin-chains.

2.2 Notations

Before writing down the spin-chain model which calculates these Wilson loop operators let us fix some notations.

The magnons can be characterized by the spectral parameters x^\pm . Introducing the parameter u , then for a general bound state of Q magnons in the physical domain they satisfy

$$x^\pm + \frac{1}{x^\pm} = \frac{u}{g} \pm \frac{iQ}{2g}, \quad g = \frac{\sqrt{\lambda}}{4\pi}. \quad (2.4)$$

For generic values of $u \in \mathbb{C}$ there are four possible solutions to the above equations with x^+ and x^- both outside the unit disk, both inside and with either one outside and the other inside.

The momentum p and energy E of the bound state are

$$e^{ip} = \frac{x^+}{x^-}, \quad E = Q + 2ig \left(\frac{1}{x^+} - \frac{1}{x^-} \right) = \sqrt{Q^2 + 16g^2 \sin^2 \frac{p}{2}}. \quad (2.5)$$

It is useful to introduce the generalized rapidity z defined on a torus with half periods

$$\omega_1 = 2\mathbb{K}, \quad \omega_2 = 2i\mathbb{K}' - 2\mathbb{K} \quad (2.6)$$

where \mathbb{K} and \mathbb{K}' are complete elliptic integral of the first kind with modulus squared $k^2 = -16g^2/Q^2$ and $k'^2 = 1 - k^2$. For real g the first period is real and the second is purely imaginary. This torus covers the four copies of the u plane. The spectral parameters, momentum and energy are expressed in terms of Jacobi elliptic functions (with modulus k) of the rapidity z

$$x^\pm(z) = \frac{Q}{4g} \left(\frac{\text{cn } z}{\text{sn } z} \pm i \right) (1 + \text{dn } z), \quad p(z) = 2 \text{am } z, \quad \sin \frac{p}{2} = \text{sn } z, \quad E(z) = Q \text{dn } z. \quad (2.7)$$

⁴The X^J state breaks $SO(5) \rightarrow SO(4)$ and would require boundary excitations. But it is not a good vacuum, as can be seen by the fact that it does not share any supersymmetry with the Wilson loop.

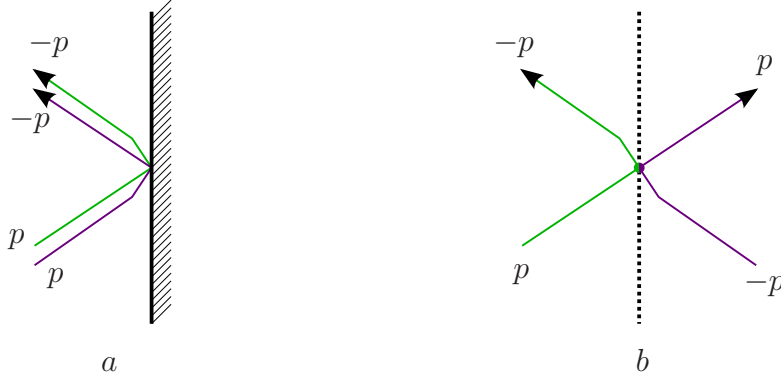


Figure 5: Using the reflection trick a semi-infinite open spin-chain can be replaced with a spin-chain on the entire line. The original magnons (a) carry representations of $\mathfrak{psu}(2|2)_R \times \mathfrak{psu}(2|2)_L$ and momentum p , which gets reflected to momentum $-p$. In the doubled picture (b) the $\mathfrak{psu}(2|2)_R$ label is carried by a magnon of momentum $-p$ on the right side, which gets scattered off the magnon of momentum p and a $\mathfrak{psu}(2|2)_L$ representation. As usual, the momentum does not get modified by the scattering and the magnon with momentum $-p$ continues on the left side, now carrying a $\mathfrak{psu}(2|2)_L$ representation

For real z both $x^\pm > 1$, the momentum is real and the energy positive. Shifting $z \rightarrow \bar{z} = z + \omega_2$ is a crossing transformation which replaces $x^\pm \rightarrow 1/x^\pm$ and reverses the signs of both the energy and momentum.

The shift $z \rightarrow \tilde{z} = z + \omega_2/2$ gives the mirror theory, where $x^- < 1 < x^+$ and both the energy and momentum are purely imaginary. In the mirror theory their roles are reversed and one defines real mirror momentum and mirror energy as

$$\tilde{p} = iE = k'Q \operatorname{sc} \tilde{z}, \quad \tilde{E} = ip = 2i \arcsin \left(\frac{Q}{4ig} \operatorname{cd} \tilde{\zeta} \right). \quad (2.8)$$

2.3 Reflections and open spin-chains

The Wilson loop breaks the $\mathfrak{psu}(2, 2|4)$ symmetry to $\mathfrak{osp}(4^*|4)$ and the symmetry preserved by the ground state of the spin-chain $\mathfrak{psu}(2|2)^2$ to the diagonal component $\mathfrak{psu}(2|2)$ with the supercharges (2.3). To write down the spin-chain system preserving this symmetry, one can use the method of images. This was done for the case of the open spin-chain associated to the D5-brane domain walls in [31].

The construction of the usual $\mathfrak{psu}(2|2)$ scattering matrix is reviewed in Appendix B, following [38].

Consider first the right boundary and a semi-infinite spin-chain to its left. Each magnon with momentum p carries a representation of $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R$. In the doubled descrip-

tion one splits each of the magnons in two, one on the original left ray with momentum p and a label of $\mathfrak{psu}(2|2)_L$ and the image magnon of momentum $-p$ and a label of $\mathfrak{psu}(2|2)_R$ on the right.

Each magnon is charged under the three central charges of the extended $\mathfrak{psu}(2|2)$: C , P and K , related also to the labels p , f and a (see Appendix B). These charges are the same for the two copies of $\mathfrak{psu}(2|2)$, and one should specify how they transform under the reflection. The reflection of the $\mathfrak{psu}(2|2)_R$ magnons to the mirror description acts as

$$\begin{aligned} p &\rightarrow -p, & x^\pm &\rightarrow -x^\mp, & a^2 &\rightarrow a^2, & f &\rightarrow fe^{ip}, \\ \bar{P} &\rightarrow -\bar{P}, & \bar{K} &\rightarrow -\bar{K}, & \bar{Q} &\rightarrow i\bar{Q}, & \bar{S} &\rightarrow -i\bar{S}. \end{aligned} \quad (2.9)$$

The extra phase acquired by f is crucial, as for neighboring magnons the two f s have to satisfy $f_1 = e^{ip_2} f_2$. For the magnons on the original (left) segment f is given by the momenta of all the magnons to their right, *up to the wall*. For consistency, the image magnons on the right, should have f which is the exponent of *minus* the momenta of all the magnons to their left up to the wall *including themselves*. The definition of f above is exactly this, after the reflection.

In formulas, enumerating the magnons on the left of the wall $1, \dots, M$, they satisfy $f_k = e^{ip_{k+1}} f_{k+1}$ with $f_M = 1$. Then continue this pattern to the full real line, where the mirror magnons are labeled $M+1, \dots, 2M$. From (2.9) $f_{M+k+1} = e^{ip_{M-k}} f_{M-k}$ so that $f_{M+k} = e^{ip_{M-k+1}} f_{M-k+1} = f_{M-k} = e^{ip_{M+k+1}} f_{M+k+1}$ satisfies the correct relation.

If one considers the two particle scattering matrix, the order of the two entries is reversed as are the signs of the momenta. Instead of f_2 it will depend on the transformed $f_1 \rightarrow e^{ip_1} f_1 = e^{ip_1+ip_2} f_2$. Indeed it is easy to check that the S-matrix (B.6), (B.7) is invariant under the transformation (2.9)

$$\mathbb{S}(e^{ip_1+ip_2} f_2, -p_2, -p_1) = \mathbb{S}(f_2, p_1, p_2). \quad (2.10)$$

The open $\mathfrak{psu}(2|2)^2$ spin-chain on the left ray is the same as a $\mathfrak{psu}(2|2)$ spin-chain on the full line. The symmetry generators in (2.3) are the diagonal components of the original $\mathfrak{psu}(2|2)_L$ and the reflection (2.9) of $\mathfrak{psu}(2|2)_R$. The scattering of the right most magnon on the left part off its image, the left most magnon on the right part is then the reflection matrix for the open spin-chain. The matrix structure of this reflection matrix is fixed by extended $\mathfrak{psu}(2|2)$ symmetry to be the same as the bulk reflection matrix

$$\mathbb{R}^{(R)aa}_{bb}(p) = R_0^{(R)}(p) \hat{\mathbb{S}}^{aa}_{bb}(p, -p). \quad (2.11)$$

where

$$\hat{\mathbb{S}}^{ab}_{cd}(p, -p) = \frac{\mathbb{S}^{ab}_{cd}(p, -p)}{S_0(p, -p)} \quad (2.12)$$

is the scattering matrix with the scalar factor removed. The explicit components of $\mathbb{R}^{(R)}(p)$ are written down in Appendix C. The scalar part of the reflection matrix is not fixed by symmetry, but it is constrained by crossing symmetry, as discussed shortly.

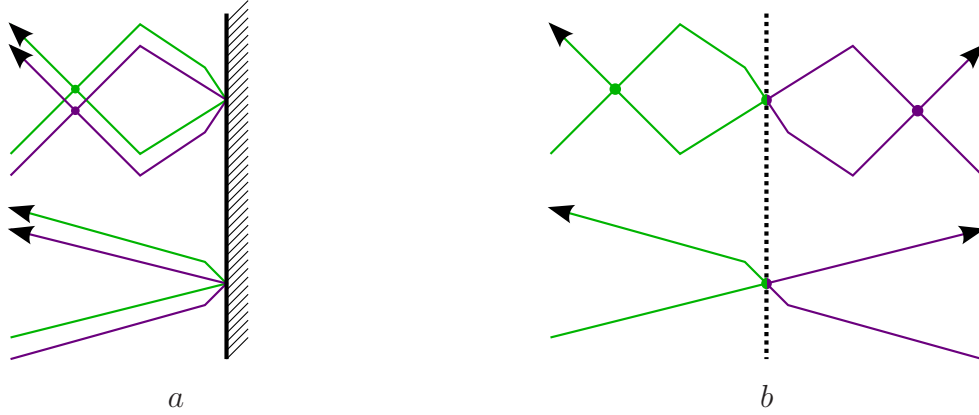


Figure 6: The boundary crossing equation equates the two processes in (a). The bottom is a single reflection. The top has a bulk scattering (a pair of $\mathfrak{psu}(2|2)$ scatterings, note that not four, since the two copies of $\mathfrak{psu}(2|2)$ interact only at the boundary), and a crossed reflection. In the doubled picture (b) this single scattering (bottom) is related to a pair of regular scatterings and one with both particles crossed.

The boundary Yang-Baxter equation is automatically satisfied, since it can be decomposed into several applications of the bulk Yang-Baxter equation for the $\mathfrak{psu}(2|2)$ chain. It was also checked explicitly in [31].

2.4 Boundary scalar factor

Following [39] it is now commonly accepted that the scattering matrix of the bulk spin-chain is invariant under crossing symmetry. The same would then be expected also for the boundary reflection. In terms of generalized rapidity variables z , the boundary crossing unitarity condition states

$$\mathbb{R}^{(R)}(z) = \mathbb{S}(z, -z) \mathbb{R}^{(R)c}(\omega_2 - z) \quad (2.13)$$

where $\mathbb{R}^{(R)c}$ is the reflection matrix in the crossed channel (see Figure 6). In our case the bulk S matrix is made of two $\mathfrak{psu}(2|2)$ matrices and including all the indices this is

$$\mathbb{R}^{(R)bb}_{aa}(z) = \mathbb{S}^{bd}_{ac}(z, -z) \mathbb{S}^{bd}_{ac}(z, -z) \mathcal{C}^{c\bar{c}} \mathcal{C}_{d\bar{d}} \mathcal{C}^{\dot{c}\bar{\dot{c}}} \mathcal{C}_{\dot{d}\bar{\dot{d}}} \mathbb{R}^{(R)\bar{d}\bar{\dot{d}}}_{\bar{c}\bar{\dot{c}}}(\omega_2 - z) \quad (2.14)$$

where $\mathcal{C}^{c\bar{c}}$ is the charge conjugation matrices.

Deriving the equation for the boundary scalar phase is actually not that hard, using the crossing invariance of the bulk scattering matrix. The crossed reflection matrix is the same as the doubly crossed $\mathfrak{psu}(2|2)$ scattering matrix. After including the usual scalar phase it

is on its own invariant under both the crossings, giving

$$\begin{aligned} \mathcal{C}^{c\bar{c}}\mathcal{C}_{d\dot{d}}\mathcal{C}^{\dot{c}\bar{c}}\mathcal{C}_{d\dot{d}}\mathbb{R}^{(R)\bar{d}\dot{d}}_{\bar{c}\dot{c}}(\omega_2 - z) &= \frac{R_0(\omega_2 - z)}{S_0(\omega_2 - z, z - \omega_2)}\mathcal{C}^{c\bar{c}}\mathcal{C}_{d\dot{d}}\mathcal{C}^{\dot{c}\bar{c}}\mathcal{C}_{d\dot{d}}\mathbb{S}^{\bar{d}\dot{d}}_{\bar{c}\dot{c}}(\omega_2 - z, z - \omega_2) \\ &= \frac{R_0(\omega_2 - z)}{S_0(\omega_2 - z, z - \omega_2)}\mathcal{C}^{\dot{c}\bar{c}}\mathcal{C}_{d\dot{d}}\mathbb{S}^{\dot{d}\bar{c}}_{\bar{c}\dot{d}}(z - \omega_2, -z) = \frac{R_0(\omega_2 - z)}{S_0(\omega_2 - z, z - \omega_2)}\mathbb{S}^{c\bar{c}}_{d\dot{d}}(-z, z). \end{aligned} \quad (2.15)$$

Expressing also the reflection matrix on the left hand side of (2.14) in terms of the bulk scattering matrix and using unitarity gives

$$R_0(z)R_0(z - \omega_2) = S_0(z, -z)S_0(z - \omega_2, \omega_2 - z). \quad (2.16)$$

This equation can be solved with $R_0(z) = S_0(z, -z)$.

This expression, though needs to be properly defined. The reason is that the solution of the crossing equation was formulated in terms of the dressing factor $\sigma(z_1, z_2)$, which is related to $S_0(z_1, z_2)$ by

$$S_0(z_1, z_2)^2 = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-} \frac{1}{\sigma(z_1, z_2)^2} \quad (2.17)$$

and in particular

$$S_0(z, -z) \sim \frac{x^+}{x^-} \sqrt{\frac{x^- + 1/x^-}{x^+ + 1/x^+}} \frac{1}{\sigma(z, -z)}. \quad (2.18)$$

So in defining $S_0(z, -z)$ one has to specify how to deal with the square root.

It is natural therefore to take an ansatz for the boundary phase as

$$R_0(z) = \frac{\sigma_B(z)}{\sigma(z, -z)} \quad (2.19)$$

such that equation (2.16) is

$$\sigma_B(z)\sigma_B(z - \omega_2) = S_0(z, -z)S_0(z - \omega_2, \omega_2 - z)\sigma(z, -z)\sigma(z - \omega_2, \omega_2 - z). \quad (2.20)$$

Using (2.18) this last equation and the unitarity constraint become

$$\sigma_B(z)\sigma_B(z - \omega_2) = \frac{x^- + 1/x^-}{x^+ + 1/x^+}, \quad \sigma_B(z)\sigma_B(-z) = 1. \quad (2.21)$$

This equation is solved in Appendix D.

The above derivation was not very careful in treating the square-root branch cut in (2.18), so there may be a sign ambiguity on the right hand side of the first equation in (2.21). The equation as written is in fact correct, to see that recall that the crossing equation was originally written for S_0 in [39] as

$$\begin{aligned} S_0(z_1 + \omega_2, z_2)S_0(z_1, z_2) &= 1/f(z_1, z_2) \\ S_0(z_1, z_2 - \omega_2)S_0(z_1, z_2) &= 1/f(z_1, z_2) \end{aligned} \quad (2.22)$$

and the dressing factor satisfies a very similar set of equations

$$\begin{aligned}\sigma(z_1 + \omega_2, z_2)\sigma(z_1, z_2) &= \frac{x_2^-}{x_2^+} f(z_1, z_2) \\ \sigma(z_1, z_2 - \omega_2)\sigma(z_1, z_2) &= \frac{x_1^+}{x_1^-} f(z_1, z_2)\end{aligned}\tag{2.23}$$

where in both cases

$$f(z_1, z_2) = \frac{x_1^- - x_2^+}{x_1^- - x_2^-} \frac{1 - 1/x_1^+ x_2^+}{1 - 1/x_1^+ x_2^-}\tag{2.24}$$

Applying the monodromies twice on (2.20) gives

$$\sigma_B(z)\sigma_B(z - \omega_2) = \left(\frac{x^-}{x^+}\right)^2 S_0(z, -z)^2 \sigma_0(z, -z)^2\tag{2.25}$$

so indeed $\sigma_B(z)$ should satisfy (2.21)

The solution found in Appendix D following the methods of [40] is

$$\sigma_B(z) = e^{i\chi_B(x^+) - i\chi_B(x^-)}\tag{2.26}$$

where

$$\chi_B(x) = -i \oint \frac{dz}{2\pi i} \frac{1}{x - z} \log \frac{\sinh 2\pi g(z + 1/z)}{2\pi g(z + 1/z)}.\tag{2.27}$$

A more general solution to the crossing equation is gotten by multiplying by the exponent of an odd function of p , so $\sigma_B e^{f_{\text{odd}}(p)}$ is also a solution. In particular the simplest modification is a linear function⁵ $\sigma_B e^{inp}$. Such a linear shift does not seem to be consistent with the one-loop calculations in [21]. All that follows assumes that this is indeed the correct boundary dressing phase, which should be verified by further direct calculations.

The leading behavior of σ_B at strong coupling can be extracted from the first term in the expansion (D.22)

$$\sigma_B(z) \approx \exp \left[2ig \left(x^+ + \frac{1}{x^+} \right) \log \left(\frac{x^+ - i}{x^+ + i} \right) - 2ig \left(x^- + \frac{1}{x^-} \right) \log \left(\frac{x^- - i}{x^- + i} \right) \right]\tag{2.28}$$

At leading order at strong coupling $x^\pm \approx e^{\pm ip/2}$, so

$$\sigma_B(p) \approx \exp \left[4ig \cos \frac{p}{2} \log \frac{1 - \sin \frac{p}{2}}{1 + \sin \frac{p}{2}} \right]\tag{2.29}$$

Together with the usual dressing factor restricted to the boundary $1/\sigma(p, -p)$ [28], the total reflection phase from the boundary at strong coupling is

$$R_0(p) \approx \exp \left[8ig \cos \frac{p}{2} \log \cos \frac{p}{2} + 4ig \cos \frac{p}{2} \log \frac{1 - \sin \frac{p}{2}}{1 + \sin \frac{p}{2}} \right].\tag{2.30}$$

⁵Recall that $x^+/x^- = e^{ip}$, so if one uses σ_B as the definition of the square root in (2.18), then indeed $S_0(z, -z) = e^{ip} \sigma_B(z)/\sigma(z, -z)$.

The analogue calculations can be done in string theory by scattering in the classical solution of [21] along the lines of [28]. In addition to the usual sine-gordon scattering along the sphere part of the sigma model one needs to include a sinh-gordon contribution from the AdS part.

The derivation of the boundary scattering phase above and in the appendix is for a fundamental magnon. For a bound state of Q magnons it is defined as $R_0^Q(z) = \sigma_B^Q(z)/\sigma^{Q,Q}(z, -z)$. It is evaluated by considering the $Q(Q-1)$ scatterings of the constituent magnons of each other and the Q reflections of the constituents. $\sigma^{Q,Q}$ is then the product of Q^2 fundamental dressing factors and is the same as the usual bulk bound state dressing factor. σ_B^Q is the product of Q fundamental boundary dressing factors which ends up being identical to (2.26) with the appropriate x^\pm for the bound state.

2.5 Twisted boundary conditions

To construct the finite size open spin-chain one has to impose boundary conditions at the left end of the chain as well. The left boundary conditions are also associated to a Wilson loop, so are very similar to the right boundary conditions.⁶ If the Wilson loop is straight, such that the angles $\theta = \phi = 0$, the boundary conditions are completely compatible, and the left reflection matrix is identical to the right reflection matrix.⁷ The resulting open spin-chain is then described as a single $\mathfrak{psu}(2|2)$ spin-chain with periodic boundary conditions and a symmetry requirement, that for every magnon of momentum p on the original segment there is another magnon with momentum $-p$ on the mirror segment, as was also the case for the $SU(2)$ sector in [21].

When the angles θ and ϕ are different from zero, the two segments of Wilson loop to which the local operator is attached are not aligned. This means that the boundary conditions are not the same, but have a relative rotation with respect to each other. In the identification of $\mathfrak{psu}(2|2)_L$ and $\mathfrak{psu}(2|2)_R$ there would be different matrices instead of the σ^3 appearing in (2.3). This can be implemented on the reflection matrix (2.11) by a $U(1)^2 \subset SU(2|2)$ rotation. For example choose the rotation to act on the fundamental representation of $\mathfrak{psu}(2|2)_R$ by

$$w_1 \rightarrow e^{i\theta} w_1, \quad w_2 \rightarrow e^{-i\theta} w_2, \quad \vartheta_1 \rightarrow e^{i\phi} \vartheta_1, \quad \vartheta_2 \rightarrow e^{-i\phi} \vartheta_2. \quad (2.31)$$

This is just the representation matrix of the spacial rotation by angle ϕ and R -rotation by angle θ . This twist matrix is labeled \mathbb{G} such that the reflection matrix from the left boundary becomes

$$\mathbb{R}^{(L)aa}_{bb}(p) = \mathbb{G}^a_c \mathbb{S}^{ac}_{db}(-p, p) \mathbb{G}^d_b. \quad (2.32)$$

⁶That is not required, of course, and it would be fun to consider open spin-chains with boundary conditions associated to different objects in the gauge theory, describing for example local insertions at the endpoint of an open Wilson loop, as in [41].

⁷The phase f also matches on this wall, since the total momentum including the magnons and their images is zero.

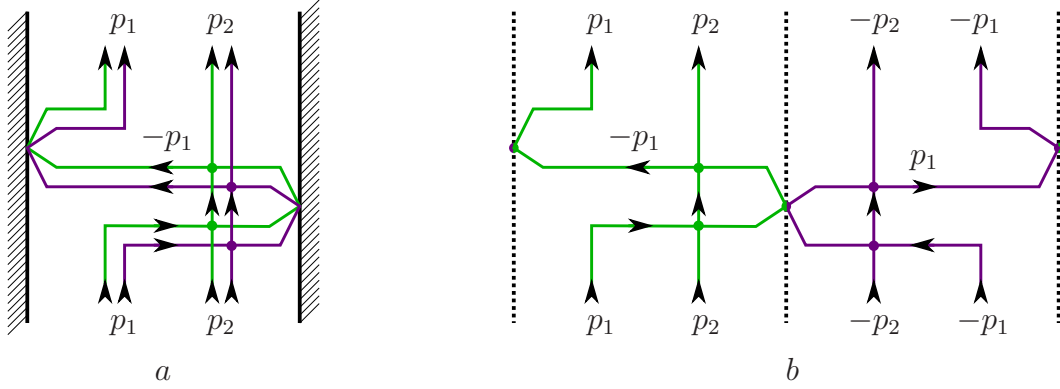


Figure 7: A graphical representation of the boundary Bethe-Yang equation in the open system with two boundaries (a), or in the doubled pictures (b) with periodic identification.

This is be very similar to the twisting arising in the cases of β and γ deformed $\mathcal{N} = 4$ SYM [42]. These models were discussed recently from the point of view of the TBA and the Y-system in [43, 44, 45, 46, 47]. The twist appears in the TBA equations as chemical potential terms.

It is now possible to write the boundary Bethe-Yang equations for this system. It states that each magnon, if scattered with the others, reflected off the right boundary, scattered back all the way to the left boundary and then back to its original position (see Figure 7), it would pick up a trivial phase. In matrix form for a spin-chain with L sites they are

$$e^{2iLp_i} = \prod_{j=i+1}^M \mathbb{S}(p_i, p_j) \mathbb{R}^{(R)}(p_i) \prod_{\substack{j=M \\ j \neq i}}^1 \mathbb{S}(p_j, -p_i) \mathbb{R}^{(L)}(-p_i) \prod_{j=1}^{i-1} \mathbb{S}(p_i, p_j). \quad (2.33)$$

As usual this set of equations can be diaganolized by using the nested Bethe ansatz equations, as was done for the case of the D5-brane defects in [31]. The set of equations for the case at hand is almost identical, one just has to insert the extra phases from \mathbb{G} , which acts diagonally on the nested equations.

In addition to the fundamental magnons the spin-chain has bound states. In the physical theory they are in totally symmetric representation of each of the $\mathfrak{psu}(2|2)$ algebras. The Q^{th} symmetric representation can be written in terms of homogeneous polynomials of degree Q in the two bosonic and two fermionic variables w_1, w_2, ϑ_3 and ϑ_4 . The representation is comprised of

$$\begin{aligned} \text{Bosons:} & \quad w_1^l w_2^{Q-l}; \quad l = 0, \dots, Q, & \quad w_1^l w_2^{Q-2-l} \vartheta_3 \vartheta_4; \quad l = 0, \dots, Q-2, \\ \text{Fermions:} & \quad w_1^l w_2^{Q-1-l} \vartheta_3; \quad l = 0, \dots, Q-1, & \quad w_1^l w_2^{Q-1-l} \vartheta_4; \quad l = 0, \dots, Q-1, \end{aligned} \quad (2.34)$$

The states of the mirror model are of more importance in what follows. They are in the totally antisymmetric representations [48], which are homogenous polynomials in a pair of fermionic and a pair of bosonic variables with the opposite labels $\vartheta_1, \vartheta_2, w_3$ and w_4 . The states and their transformation under \mathbb{G} are

Bosons:

$$\begin{aligned} w_3^l w_4^{Q-l} &\rightarrow e^{i(2l-Q)\phi} w_3^l w_4^{Q-l}; & l = 0, \dots, Q, \\ w_3^l w_4^{Q-2-l} \vartheta_1 \vartheta_2 &\rightarrow e^{i(2l+2-Q)\phi} w_3^l w_4^{Q-2-l} \vartheta_1 \vartheta_2; & l = 0, \dots, Q-2, \end{aligned} \quad (2.35)$$

Fermions:

$$\begin{aligned} w_3^l w_4^{Q-1-l} \vartheta_1 &\rightarrow e^{i(2l+1-Q)\phi+i\vartheta} w_3^l w_4^{Q-1-l} \vartheta_1; & l = 0, \dots, Q-1, \\ w_3^l w_4^{Q-1-l} \vartheta_2 &\rightarrow e^{i(2l+1-Q)\phi-i\vartheta} w_3^l w_4^{Q-1-l} \vartheta_2; & l = 0, \dots, Q-1, \end{aligned}$$

Then it is easy to calculate the supertrace

$$\begin{aligned} \text{sTr}_Q \mathbb{G} &= \sum_{l=0}^Q e^{(2l-Q)i\phi} + \sum_{l=0}^{Q-2} e^{(2l+2-Q)i\phi} - 2 \cos \theta \sum_{l=0}^{Q-1} e^{(2l+1-Q)i\phi} \\ &= 2(\cos \phi - \cos \theta) \sum_{l=0}^{Q-1} e^{(2l+1-Q)i\phi} = 2(\cos \phi - \cos \theta) \frac{\sin Q\phi}{\sin \phi} \end{aligned} \quad (2.36)$$

3 Wrapping corrections

An elegant way to formulate the Bethe-Yang equations is in terms of the transfer matrix, capturing the monodromy around the spin-chain. In the case of open spin-chains the analog quantity is known as the double-row transfer matrix [49] defined as

$$\mathbb{T}(z|z_1, \dots, z_M) = \text{sTr} \left[\mathbb{S}(z, z_1) \cdots \mathbb{S}(z, z_M) \mathbb{R}^{(R)}(z) \mathbb{S}(z_M, -z) \cdots \mathbb{S}(z_1, -z) \mathbb{R}^{(L)c}(z - \omega_2) \right] \quad (3.1)$$

where the trace is performed only over the states of the magnon with generalized rapidity z and it carries the matrix indices of all the other magnons. This is illustrated in Figure 8.

In terms of the transfer matrix the Bethe-Yang equations take the very simple form $\mathbb{T}(z_i|z_1, \dots, z_M) = -1$.

Within the context of integrability of $\mathcal{N} = 4$ SYM the double row transfer matrix was used in calculating finite size corrections to the spectrum of insertions into determinant operators (giant gravitons in the dual string theory) in [50, 51], generalizing the pioneering work of Bajnok-Janik [34]. Here the same is done for the open spin-chain model related to Wilson loops.

It is easy to show that in the case studied here, where the reflection matrix is proportional to the bulk scattering matrix and commutes with the twist matrix, the transfer matrix factorizes to the product of two twisted transfer matrices of $\mathfrak{psu}(2|2)$ as in Figure 9. Instead of

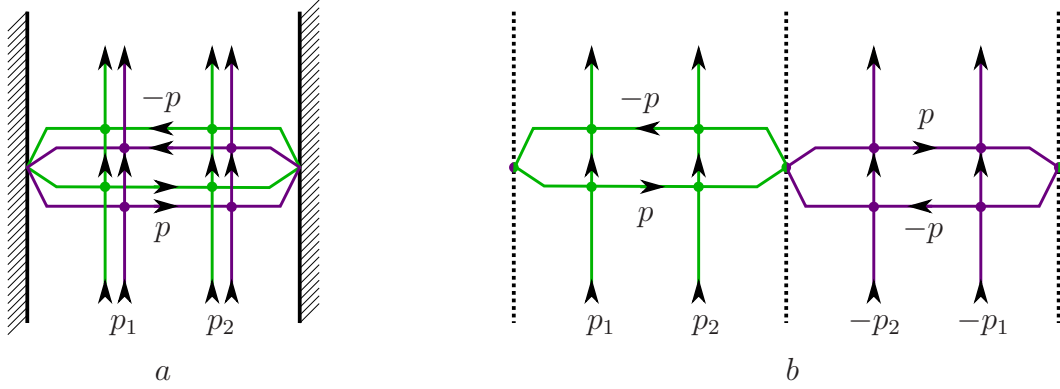


Figure 8: The double-row transfer matrix evaluated on a state with two physical magnons of momentum p_1 and p_2 (a). The auxiliary magnon, of momentum p scatters off them, reflects off the boundary, scatters again and then does a crossed reflection back to the original starting point, where it can be traced. In the doubled version (b) there are a pair of auxiliary $\mathfrak{psu}(2|2)$ magnons of momentum p and $-p$. They scatter off all the physical magnons and their images as well as off each other. Once in the forward direction and once doubly crossed.

writing formulas with intractable indices, the reader should be convinced by Figure 10. The twisted transfer matrices of the closed spin-chain were studied in [45], see also [43, 44, 46, 47].

The main example of interest is the ground state energy, so we can omit all the magnons except for the auxiliary one being traced over. The result is a transfer function $T_Q^{\phi,\theta}(z)$ for an auxiliary bound state of Q magnons with rapidity z and with the twist angles explicitly spelled out. Using $\mathbb{R}^{(L)}(z) = \mathbb{G} \mathbb{R}^{(R)}(-z) \mathbb{G}$ and the same manipulations as in (2.15) and (2.20) one finds

$$\begin{aligned}
T_Q^{\phi,\theta}(z) &= \mathbb{T}(z) = \text{sTr} \left[\mathbb{R}^{(R)}(z) \mathbb{R}^{(L)c}(z - \omega_2) \right] = \text{sTr} \left[\mathbb{R}^{(R)}(z) \mathbb{G} \mathbb{R}^{(R)c}(-z + \omega_2) \mathbb{G} \right] \\
&= \frac{R_0(z) R_0(-z + \omega_2)}{S_0(z, -z) S_0(-z + \omega_2, z - \omega_2)} \mathbb{S}_{aa}^{bb}(z, -z) \mathbb{G}_b^c \mathbb{S}_{cb}^{ad}(-z, z) \mathbb{G}_d^a \\
&= \sigma_B(z) \sigma_B(-z + \omega_2) \left(\frac{x^-}{x^+} \right)^2 (\text{sTr} \mathbb{G})^2
\end{aligned} \tag{3.2}$$

The final expression is very simple because the twist matrices \mathbb{G} commute with the scattering matrix and thus lead to the factorization into two traces of the twist matrix and a scalar phase.

The auxiliary particle can be in any representation. For the purpose of calculating the wrapping effects it should be a bound state in the mirror model, which are in the completely antisymmetric representation. Each of the supertraces is performed in the Q^{th} antisymmetric

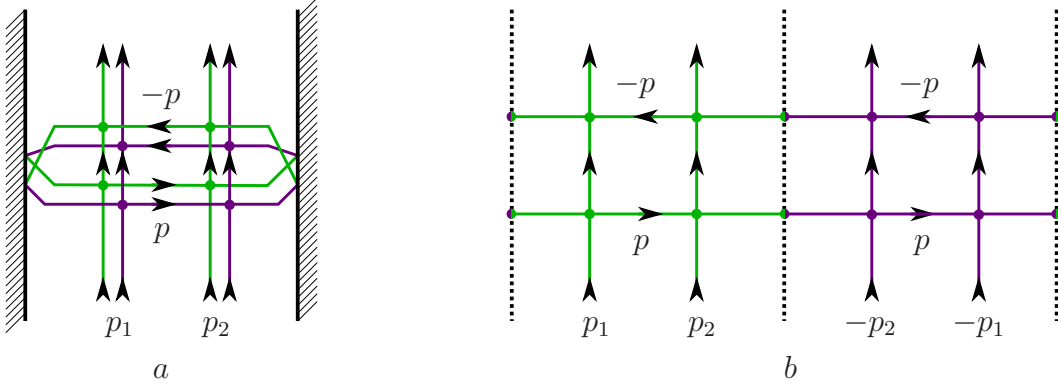


Figure 9: An alternative description of the double row transfer matrix, see the sequence of steps in Figure 10. The result in the doubled picture (b) is a pair of regular $\mathfrak{psu}(2|2)$ transfer matrices. The only remnant of the boundary reflections is the phase factor σ_B at each boundary and possibly a twist matrix. In the open spin-chain picture (a) the boundary reflection is completely diagonal.

representation of $\mathfrak{psu}(2|2)$ which using (2.36) leads to

$$(\text{sTr}_Q \mathbb{G})^2 = 4(\cos \phi - \cos \theta)^2 \frac{\sin^2 Q \phi}{\sin^2 \phi} \quad (3.3)$$

The ground state Z^J has classical dimension $\Delta = J$. The first correction comes from the leading wrapping effect, which is

$$\delta E \approx -\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_0^{\infty} d\tilde{p} \log \left(1 + T_Q^{(\phi, \theta)} \left(z + \frac{\omega_2}{2} \right) e^{-2L\tilde{E}_Q} \right) \quad (3.4)$$

Here $\tilde{z} = z + \omega_2/2$ is the generalized rapidity for a magnon of the mirror theory of momentum \tilde{p} . L is the length of the world-sheet and is equal to the number of sites J on the spin-chain. Finally $\tilde{E}_Q = \log x^+/x^-$ is the energy of the bound state in the mirror theory. A similar expression arises from the full thermodynamic Bethe ansatz treatment of the problem, where \tilde{E}_Q is replaced by $\tilde{\epsilon}_Q$, the solution of the TBA equations, accounting for entropy, in addition to the energy. In that case L is interpreted as the inverse temperature. The $(x^-/x^+)^2$ term in (3.2) can be absorbed in a shift $L \rightarrow L + 1$

The contribution from the boundary dressing phase is

$$\sigma_B \left(z + \frac{\omega_2}{2} \right) \sigma_B \left(-z + \frac{\omega_2}{2} \right) = \frac{\sigma_B \left(z + \frac{\omega_2}{2} \right)}{\sigma_B \left(z - \frac{\omega_2}{2} \right)} = e^{i\chi_B(x^+(z)) - i\chi_B(1/x^-(z)) - i\chi_B(1/x^+(z)) + i\chi_B(x^-(z))} \quad (3.5)$$

since a shift by $\omega_2/2$ replaces $x^- \rightarrow 1/x^-$ and by $-\omega_2/2$ replaces $x^+ \rightarrow 1/x^+$. For real z one has $|x^\pm(z)| > 0$ and to define $\chi_B(1/x^\pm(z))$ one should use the monodromy property (D.9),

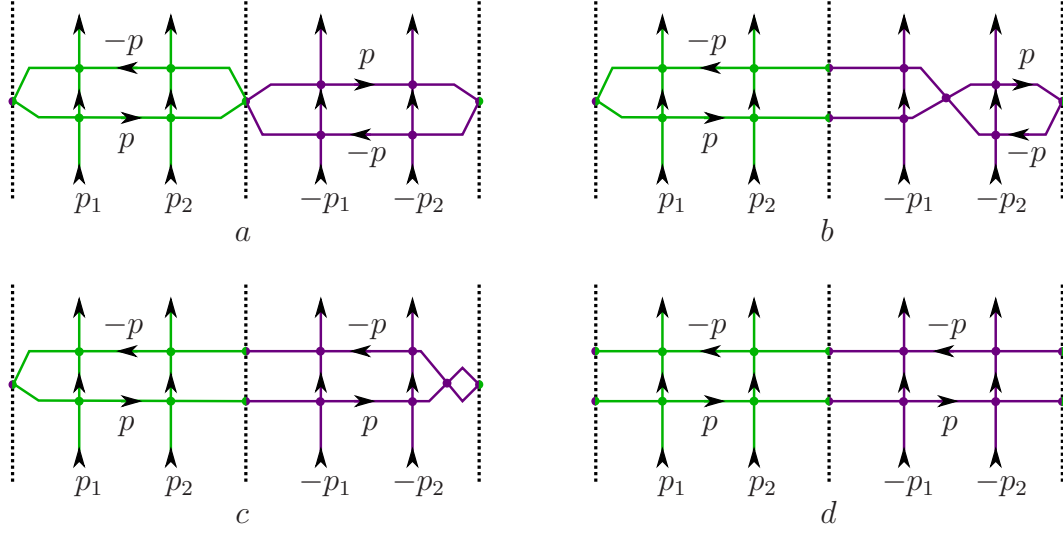


Figure 10: The sequence of Yang-Baxter and crossing/unitarity relations $a \rightarrow b \rightarrow c \rightarrow d$ used to relate Figure 8 and Figure 9. Recall that up to a scalar phase and a twist with which it commutes, the reflection matrix is the same as the regular $\mathfrak{psu}(2|2)$ scattering matrix.

this gives

$$\begin{aligned} \frac{\sigma_B(z + \frac{\omega_2}{2})}{\sigma_B(z - \frac{\omega_2}{2})} &= e^{2i(\chi_B(x^+) + \chi_B(x^-))} \frac{(2\pi g)^2 (x^+ + 1/x^+)(x^- + 1/x^-)}{\sinh(2\pi g(x^+ + 1/x^+)) \sinh(2\pi g(x^- + 1/x^-))} \\ &= e^{2i(\chi_B(x^+) + \chi_B(x^-))} \frac{(2\pi)^2 (u^2 + Q^2/4)}{\sinh^2(2\pi u)} \end{aligned} \quad (3.6)$$

Where the last equality was written using the relation $g(x^\pm + 1/x^\pm) = u \pm iQ/2$ and the periodicity of the hyperbolic sine function. Note that the dressing phase contribution has a pole at $u = 0$, which is also $\tilde{p} = 0$, representing the contribution of zero momentum states for all values of Q .

Usually one expands the log function in (3.4) to linear order and integrates over \tilde{p} . In the case at hand this approximation is invalid, since $T_Q^{\phi, \theta}(\tilde{p})$ has the double pole at $\tilde{p} = 0$, and thus is not uniformly small for all \tilde{p} (even for small g or for large L). Instead write [52, 53]

$$T_Q = \frac{T_Q^{\text{pole}}}{\tilde{p}^2} + T_Q^{\text{reg}}, \quad (3.7)$$

and use

$$\int_0^\infty d\tilde{p} \log \left(1 + \frac{c^2}{\tilde{p}^2} \right) = \pi c, \quad (3.8)$$

to get

$$\begin{aligned}
\delta E &= -\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_0^{\infty} d\tilde{p} \log \left(1 + \frac{T_Q^{\text{pole}}}{\tilde{p}^2} e^{-2L\tilde{E}_Q} \right) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_0^{\infty} d\tilde{p} \log \left(\frac{1 + T_Q e^{-2L\tilde{E}_Q}}{1 + T_Q^{\text{pole}} e^{-2L\tilde{E}_Q}/\tilde{p}^2} \right) \\
&= -\frac{1}{2} \sum_{Q=1}^{\infty} \sqrt{T_Q^{\text{pole}} e^{-2L\tilde{E}_Q}} - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_0^{\infty} d\tilde{p} \frac{T_Q^{\text{reg}} e^{-2L\tilde{E}_Q}}{1 + T_Q^{\text{pole}} e^{-2L\tilde{E}_Q}/\tilde{p}^2} + \dots
\end{aligned} \tag{3.9}$$

The pole term dominates at large L and at weak coupling, since it comes with $e^{-L\tilde{E}_Q}$ rather than $e^{-2L\tilde{E}_Q}$. And it is given by the simple expression

$$\sqrt{T_Q^{\text{pole}} e^{-2L\tilde{E}_Q}} = 2 \frac{\cos \phi - \cos \theta}{\sin \phi} \sin Q\phi \operatorname{res}_{\tilde{p} \rightarrow 0} \left[e^{i(\chi_B(x^+) + \chi_B(x^-))} \frac{2\pi \sqrt{u^2 + Q^2/4}}{(-1)^Q \sinh(2\pi u)} \left(\frac{x^-}{x^+} \right)^{L+1} \right] \tag{3.10}$$

One may be concerned about the choice of sign on the right hand side. If the integral in (3.8) is regarded as a real integral of a positive definite quantity, then one should choose the positive branch of $\sqrt{T_Q^{\text{pole}} e^{-2L\tilde{E}_Q}}$. It is actually more natural to take it to be an analytic expression and then the hyperbolic sine in the denominator is $\sinh(2\pi(u \pm iQ/2))$. See a careful discussion of such signs in [53].

By using the explicit expressions at small \tilde{p}

$$\begin{aligned}
u &= \frac{\tilde{p}}{2} \sqrt{1 + \frac{16g^2}{\tilde{p}^2 + Q^2}} = \frac{\tilde{p}}{2} \sqrt{1 + \frac{16g^2}{Q^2}} + O(\tilde{p}^3) \\
e^{-\tilde{E}_Q} &= \frac{x^-}{x^+} = \frac{\sqrt{1 + \frac{16g^2}{\tilde{p}^2 + Q^2}} - 1}{\sqrt{1 + \frac{16g^2}{\tilde{p}^2 + Q^2}} + 1} = \frac{\sqrt{1 + \frac{16g^2}{Q^2}} - 1}{\sqrt{1 + \frac{16g^2}{Q^2}} + 1} + O(\tilde{p}^2)
\end{aligned} \tag{3.11}$$

equation (3.10) becomes

$$\sqrt{T_Q^{\text{pole}} e^{-2L\tilde{E}_Q}} = 2 \frac{\cos \phi - \cos \theta}{\sin \phi} \sin Q\phi \frac{(-1)^Q Q}{\sqrt{1 + \frac{16g^2}{Q^2}}} \left(\frac{\sqrt{1 + \frac{16g^2}{Q^2}} - 1}{\sqrt{1 + \frac{16g^2}{Q^2}} + 1} \right)^{L+1} e^{i(\chi_B(x^+) + \chi_B(x^-))} \tag{3.12}$$

At weak coupling the dressing phase starts contributing at order g^4 , so it can be ignored when considering the first two terms

$$\sqrt{T_Q^{\text{pole}} e^{-2L\tilde{E}_Q}} = 2 \frac{\cos \phi - \cos \theta}{\sin \phi} \sin Q\phi (-1)^Q \left[\frac{(4g^2)^{L+1}}{Q^{2L+1}} - 2(L+2) \frac{(4g^2)^{L+2}}{Q^{2L+3}} + O(g^{2(L+3)}) \right] \tag{3.13}$$

And then from (3.9), the leading contribution to the ground state energy at small g is

$$\begin{aligned}
\delta E &\approx -(4g^2)^{L+1} \frac{\cos \phi - \cos \theta}{\sin \phi} \sum_{Q=1}^{\infty} \frac{(-1)^Q \sin Q\phi}{Q^{2L+1}} \\
&= -\frac{(4g^2)^{L+1}}{2i} \frac{\cos \phi - \cos \theta}{\sin \phi} (\text{Li}_{2L+1}(-e^{i\phi}) - \text{Li}_{2L+1}(-e^{-i\phi})) \\
&= -\frac{(-16\pi^2 g^2)^{L+1}}{4\pi(2L+1)!} \frac{\cos \phi - \cos \theta}{\sin \phi} B_{2L+1} \left(\frac{\phi}{2\pi} + \frac{1}{2} \right)
\end{aligned} \tag{3.14}$$

Where B_{2L+1} are Bernoulli polynomials. For $L = 0$ it is $B_1(x) = x - \frac{1}{2}$, so

$$E = \delta E = 2g^2 \frac{\cos \phi - \cos \theta}{\sin \phi} \phi + O(g^4) \tag{3.15}$$

This is the same as the one loop perturbative calculation, equation (1.5), as calculated originally in [15].

For $L > 0$ this calculates the leading correction to the energy of the ground state Z^L in the spin-chain, which happens at order $L + 1$ in perturbation theory and should hold up to order $2L + 2$, where double wrapping and the second term in (3.9) start to contribute. For $L > 0$ this would involve gauge theory graphs like in Figure 4d, which so far not been calculated directly.

For large L (scaled with g) this should give the correct answer to all values of the coupling and match with classical string solutions similar to those in [21].

4 The twisted boundary TBA

The calculation so far enabled the derivation of the one-loop result for the generalized quark-antiquark potential. To find the answer at all values of the coupling requires to solve exactly for the ground state energy of the open spin-chain with twisted boundary conditions. This can be done by using the boundary thermodynamic Bethe ansatz (BTBA) equations [54].

The idea is to exchange the space and time directions and instead of calculating the partition function (or Witten index) of the finite size system over an infinite time, to calculate the partition function of the theory on an infinite circle over a finite time [55]. In the case of a periodic model the result is a thermal partition function

$$Z = \text{Tr} e^{-R H_L} = \text{Tr} e^{-H_R^m/T}, \quad T = 1/L, \tag{4.1}$$

where H_L is the hamiltonian for the original model on the interval of width L and H_R^m is the hamiltonian of the mirror model on the interval of width R .

In the case of the open spin-chain the boundaries get replaced with initial and final boundary states $|\mathcal{B}_\alpha\rangle$ and one calculates the transition amplitude between boundaries of

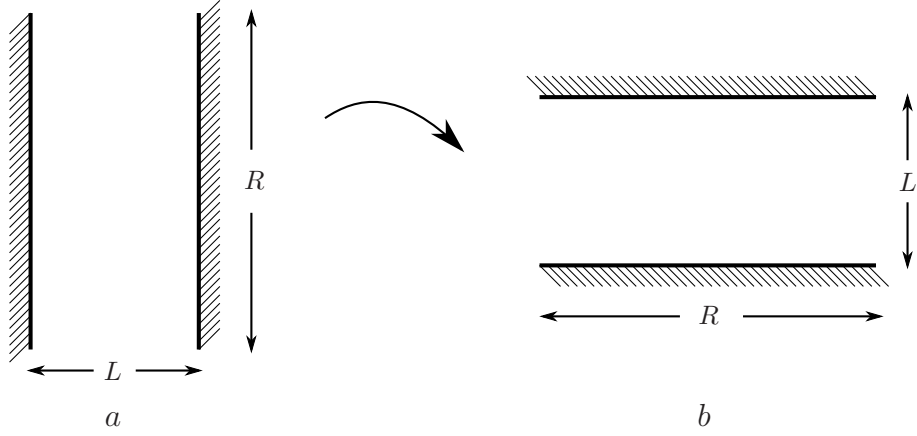


Figure 11: The boundary TBA: In the original setup (a) one has to find the exact ground state spectrum for finite size L . Instead, in the BTBA approach (b) one considers the system on a very large cylinder of circumference R , where the spectrum can be evaluated exactly and propagates a boundary state over a finite time L .

type α, β

$$Z_{\alpha, \beta} = \text{Tr} e^{-R H_{L(\alpha, \beta)}} = \langle \mathcal{B}_\beta | e^{-H_R^m / T} | \mathcal{B}_\alpha \rangle, \quad T = 1/L, \quad (4.2)$$

see Figure 11. In general there are many different boundary conditions for the open spin-chain related to strings ending on different objects in $AdS_5 \times S^5$. The ones addressed here are those related to insertions into Wilson loop operators as discussed in the previous sections and the two boundaries are identical up to the rotation by the angles ϕ and θ .

The boundary state $|\mathcal{B}\rangle$ is a superposition of states with different numbers of magnons of arbitrary momenta, subject to the symmetry under $\tilde{p} \rightarrow -\tilde{p}$. The amplitude for emitting a pair of magnons by the boundary is represented by the matrix $K(\tilde{p})$ which is closely related to the reflection matrix R . It is the charge conjugate of the reflection matrix analytically continued to the mirror kinematics⁸

$$K^{a\dot{a}bb}(\tilde{z}) = \mathcal{C}^{ac} \mathcal{C}^{\dot{a}\dot{c}} R_{c\dot{c}}^{bb}(-\tilde{z} + \frac{\omega_2}{2}), \quad (4.3)$$

where \mathcal{C} is the charge conjugation matrix.

In addition, the boundary state can have single magnon contributions subject to the constraint $\tilde{p} = 0$. These arise when the scattering matrix has a double pole at $\tilde{p} = 0$, which is indeed the case here. The amplitude for the emission of such a magnon is $g^{a\dot{a}}$, related to K by [56]

$$\text{Res}_{\tilde{p} \rightarrow 0} K^{a\dot{a}bb}(\tilde{p}) = 2i g^{a\dot{a}} g^{bb} \quad (4.4)$$

⁸In this section quantities are written in terms of real mirror rapidity \tilde{z} .

With these two quantities, and under the usual assumptions of integrability, the boundary state can be written as [57]

$$|\mathcal{B}\rangle = \mathcal{N} \left(1 + g^{a\dot{a}} A_{a\dot{a}}^\dagger(0) \right) \exp \left[\int_0^{\omega_1/2} d\tilde{z} K^{a\dot{a}bb}(\tilde{z}) A_{a\dot{a}}^\dagger(-\tilde{z}) A_{bb}^\dagger(\tilde{z}) \right] |0\rangle \quad (4.5)$$

where $A_{a\dot{a}}^\dagger(\tilde{p})$ is a creation operator for a magnon with quantum numbers a and \dot{a} in $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R$ and \mathcal{N} is a normalization constant. This expression was written in terms of “out states”, where the momentum of the right magnon is larger than the left magnon. In terms of “in states” the same expression can be written by virtue of the boundary crossing relation (2.14) as

$$|\mathcal{B}\rangle = \mathcal{N} \left(1 + g^{a\dot{a}} A_{a\dot{a}}^\dagger(0) \right) \exp \left[\int_0^{\omega_1/2} d\tilde{z} K^{a\dot{a}bb}(-\tilde{z}) A_{a\dot{a}}^\dagger(\tilde{z}) A_{bb}^\dagger(-\tilde{z}) \right] |0\rangle \quad (4.6)$$

Note that the in addition to fundamental magnons these expressions also includes bound states of the mirror model. If one wishes to be more explicit about that, it is possible to add an index Q to g , K and A and then sum over Q .

The difference between the initial state and the final state, which can be labeled $|\mathcal{B}^{(R)}\rangle$ and $|\mathcal{B}^{(L)}\rangle$ is the same as the difference between the respective scattering matrices — it is a global rotation.

Indeed, as mentioned before, when the two boundaries are of the type associated to the Wilson loops, then by repeated use of the Yang-Baxter equation and unitarity, it is possible to replace in all calculations the reflection matrix by a diagonal matrix with only the scalar phase and the twist. Then one gets

$$\begin{aligned} K^{(R) a\dot{a}bb}(\tilde{z}) &= \mathcal{C}^{ac} \delta_c^{\dot{b}} \mathcal{C}^{\dot{a}c} \delta_c^b \sigma_B(-\tilde{z} + \frac{\omega_2}{2}) \\ \bar{K}^{(L) a\dot{a}bb}(\tilde{z}) &= \mathcal{C}_{ac} \mathbb{G}_b^c \mathcal{C}_{\dot{a}c} \mathbb{G}_b^{\dot{c}} \sigma_B(-\tilde{z} + \frac{\omega_2}{2}) \end{aligned} \quad (4.7)$$

where \bar{K} is the conjugate of K , the amplitude of absorption of a pair of magnons into the final state. Since the order of the operators is reversed, in the “in states” basis the amplitude is $\bar{K}(\tilde{z})$ and in the “out states” basis it is $\bar{K}(-\tilde{z})$.

The vacuum transfer matrix $T_Q^{\phi,\theta}$ is then the inner product of this two-particle contribution

$$\begin{aligned} T_Q^{\phi,\theta} &= \bar{K}_{a\dot{a}bb}^{(L)}(-\tilde{z}) K^{(R) b\dot{b}a\dot{a}}(\tilde{z}) = \mathcal{C}_{ad} \mathbb{G}_b^d \mathcal{C}_{\dot{a}d} \mathbb{G}_b^{\dot{d}} \mathcal{C}^{ac} \delta_c^{\dot{b}} \mathcal{C}^{\dot{a}c} \delta_c^b \sigma_B(\tilde{z} + \frac{\omega_2}{2}) \sigma_B(-\tilde{z} + \frac{\omega_2}{2}) \\ &= (\text{sTr } \mathbb{G})^2 \sigma_B(\tilde{z} + \frac{\omega_2}{2}) \sigma_B(-\tilde{z} + \frac{\omega_2}{2}) = (\text{sTr } \mathbb{G})^2 \lambda_Q(z) \end{aligned} \quad (4.8)$$

where after absorbing the factor of $(x^-/x^+)^2$ in (3.2) by the shift $L \rightarrow L+1$, λ_Q is given by (3.6)⁹

$$\lambda_Q = e^{2i(\chi_B(x^+) + \chi_B(1/x^-))} \frac{(2\pi)^2(u^2 + Q^2/4)}{\sinh^2(2\pi u)} \quad (4.9)$$

⁹Since mirror kinematics, where $|x^-| < 1$, are used in this section, then compared to the notations in the previous section one replaces $x^- \rightarrow 1/x^-$.

A crucial ingredient in deriving the result in Section 3 is the fact that the scattering matrix has a pole at zero mirror momentum, seen here in λ_Q . This represents contributions from single zero momentum states and the factor $g^{a\dot{a}}$ in equation (4.5) is related to the residue. For a generic bound state of Q magnons it is

$$g_Q^{a\dot{a}} = \mathcal{C}^{a\dot{a}}(-1)^Q \frac{Q}{\sqrt{1 + \frac{16g^2}{Q^2}}} \frac{\sqrt{1 + \frac{16g^2}{Q^2}} - 1}{\sqrt{1 + \frac{16g^2}{Q^2}} + 1}, e^{i\chi_B(x^+) + i\chi_B(1/x^-)} \quad (4.10)$$

The derivation of the boundary TBA equations follows the standard procedure [54] of expressing the boundary state as a sum over eigenstate of the mirror hamiltonian, introducing densities for particles and holes and minimizing the free energy, accounting for the entropy of the states. As explained, the boundary state is completely symmetric in the exchange of $\tilde{p} \rightarrow -\tilde{p}$ and under the replacement of $\mathfrak{psu}(2|2)_R$ and $\mathfrak{psu}(2|2)_L$. The density of momentum carrying particles can therefore be defined for positive \tilde{p} only, representing a pair of particles (or holes) at momentum \tilde{p} and $-\tilde{p}$. At the nested level the replacement of $u \rightarrow -u$ is accompanied by replacement of the right and left groups.

The mirror theory is exactly the same as in the case of periodic spin-chains and the Bethe equations (which lead to the kernels in the TBA equations) are also the same, except that one has to account for the symmetry of the states overlapping with the boundary state.

The resulting BTBA equations are therefore the same as for the closed spin-chains [58, 59, 60] with the following modifications:

1. The momentum carrying densities ϵ_Q are defined only for positive u .
2. Since now the ϵ_Q densities represent pairs of particles of opposite momentum, all kernels coupling to them are doubled $K^{QQ'}(u, u') \rightarrow K^{QQ'}(u, u') + K^{QQ'}(u, -u')$. The integration domain for u' is $[0, \infty)$.
3. All other densities are symmetric under

$$\epsilon_{y^\pm}^{(+)}(u) = \epsilon_{y^\pm}^{(-)}(-u), \quad \epsilon_{vw|M}^{(+)}(u) = \epsilon_{vw|M}^{(-)}(-u), \quad \epsilon_{w|M}^{(+)}(u) = \epsilon_{w|M}^{(-)}(-u). \quad (4.11)$$

4. The twisting matrix \mathbb{G} introduces chemical potentials, as in the case of the usual close spin-chain twisted TBA. See [43, 44, 45].
5. There is an extra driving term from the boundary dressing phase contributing to the equation for ϵ_Q . This is equal to $-\log \lambda_Q$.
6. The ground state energy is

$$\delta E = -\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_0^{\infty} du \frac{d\tilde{p}}{du} \log(1 + \lambda_Q e^{-\epsilon_Q}) \quad (4.12)$$

An alternative formulation extends the definition of ϵ_Q to negative u by $\epsilon_Q(-u) = \epsilon_Q(u)$ and then integrate u over $(-\infty, \infty)$ with the regular kernels. In this formulation the BTBA equations are identical to the usual twisted periodic TBA equations except for the extra driving term $-\log \lambda_Q$ for Y_Q . In the notations of [60] the equations are

$$\begin{aligned}
\log Y_{w|M}^{(a)} &= -2i\theta + \log \left(1 + 1/Y_{w|N}^{(a)} \right) \star K_{NM} + \log \frac{1 + 1/Y_-^{(a)}}{1 + 1/Y_+^{(a)}} \hat{\star} K_M \\
\log Y_{vw|M}^{(a)} &= -2i\phi + \log \left(1 + 1/Y_{vw|N}^{(a)} \right) \star K_{NM} + \log \frac{1 + 1/Y_-^{(a)}}{1 + 1/Y_+^{(a)}} \hat{\star} K_M - \log (1 + Y_Q) \star K_{xy}^{QM} \\
\log Y_{\pm}^{(a)} &= -i(\phi - \theta) - \log (1 + Y_Q) \star K_{\pm}^{QM} + \log \frac{1 + 1/Y_{vw|M}^{(a)}}{1 + 1/Y_{w|M}^{(a)}} \star K_M \\
\log Y_Q &= -2iQ\phi - (L+1)\tilde{E}_Q - \log \lambda_Q + \log (1 + Y_{Q'}) \star K_{sl(2)}^{Q'Q} \\
&\quad + \sum_{a=\pm} \left[\log \left(1 + 1/Y_{vw|M}^{(a)} \right) \star K_{vwx}^{MQ} + \sum_{\pm} \log \left(1 + 1/Y_{\pm}^{(a)} \right) \hat{\star} K_{\pm}^{yQ} \right]
\end{aligned} \tag{4.13}$$

with

$$Y_Q = e^{-\epsilon_Q}, \quad Y_{vw|M}^{(a)} = e^{\epsilon_{vw|M}^{(a)}}, \quad Y_{w|M}^{(a)} = e^{\epsilon_{w|M}^{(a)}}, \quad Y_{y\pm}^{(a)} = e^{\epsilon_{y\pm}^{(a)}}. \tag{4.14}$$

The definitions of the convolutions and all the kernels are given in [60] (though note the slightly different conventions from this manuscript).

The energy is then given by

$$E = -\frac{1}{4\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d\tilde{p}}{du} \log(1 + Y_Q). \tag{4.15}$$

One can derive the simplified (and hybrid) TBA equations as usual by acting with the inverse kernel $(K+1)_{NM}^{-1}$ (see [60]), which removes the λ_Q terms from the equations,¹⁰ due to the relation $\lambda_{Q-1}(u)\lambda_{Q+1}(u) = \lambda_Q(u-i/2)\lambda_Q(u+i/2)$. The derivation of the Y -system equations [61] then follows as usual [60], and they are identical to the regular equations for the spectral problem in AdS/CFT .

Assuming $Y_Q = 0$ one finds the asymptotic solution for the auxiliary particles which is identical to the case of twisted periodic TBA [47]

$$\begin{aligned}
Y_{w|M}^{(\pm)\circ} &= \frac{\sin M\theta \sin(M+2)\theta}{\sin^2 \theta}, & Y_{vw|M}^{(\pm)\circ} &= \frac{\sin M\phi \sin(M+2)\phi}{\sin^2 \phi}, \\
Y_+^{(\pm)\circ} &= \frac{\cos \phi}{\cos \theta}, & Y_-^{(\pm)\circ} &= \frac{\cos \phi}{\cos \theta},
\end{aligned} \tag{4.16}$$

¹⁰Except possibly for the usual subtleties in the equation for Y_1 .

Feeding this back into the equation for Y_Q and including the boundary driving term gives

$$Y_Q^\circ = 4(\cos \phi - \cos \theta)^2 \frac{\sin^2 Q\phi}{\sin^2 \phi} \lambda_Q e^{-(L+1)\tilde{E}_Q} \quad (4.17)$$

Note that the assumption of small Y_Q is true only for $u \neq 0$ and this is not a good approximate solution at $u = 0$. Still as shown in the previous section, the pole contribution does give the correct one loop gauge theory result.

5 Discussion

This paper presents the first step in finding the exact quark–antiquark potential in $\mathcal{N} = 4$ SYM: The set of twisted boundary TBA equations whose solution is conjectured to give the desired function. As explained in the introduction, with the extra two parameters θ and ϕ the generalized potential is the same as the generalized cusp anomalous dimension.

The resulting quantity gives the conformal dimension of cusps in arbitrary Maldacena-Wilson loops in this theory. The discussion here has been on the most symmetric cases, of the antiparallel lines or infinite cusp. But the divergences are a UV property and depend only on the local structure of the Wilson loop. Knowledge of the exact generalized cusp anomalous dimension allows therefore to renormalize any Wilson loop (with the usual scalar coupling) in this theory with arbitrary number of cusps. One follows usual normalization, now that the UV behavior near the cusp is under control, see [11].

The discussion here was for real angle ϕ as appropriate for the Euclidean theory (or a Euclidean cusp in the Lorentzian theory). There is no reason to restrict to that. All the expression are analytic in ϕ and a Wick-rotation $\phi = i\varphi$ gives the cusp anomalous dimension for an arbitrary boost angle φ . At large φ the result should be proportional to φ , with the coefficient being a quarter of the universal cusp anomalous dimension $\gamma_{\text{cusp}}/4$. In particular for large φ the BES equation for the cusp anomalous dimension [62] (and its solution [63]) should be recovered from the twisted boundary TBA equation.

For large but finite φ this gives the regularized cusp anomalous dimension which plays a role in scattering amplitudes in $\mathcal{N} = 4$ SYM when imposing a Higgs-VEV regulator as in [64, 65, 7].

The TBTBA equations contain much more information than that, though. As usual, by contour deformation one can calculate the exact spectrum of excited states of the relevant string (or cusped Wilson loops with local operator insertions). For large L one can approximate the solution by solving the boundary Bethe-Yang equations instead.

In the case of large imaginary angle, these are excitation of the string with lightlike cusp describing scattering amplitudes in AdS . This spectrum is a crucial ingredient in the OPE approach to lightlike Wilson loops as discussed in [66, 67] and should be closely related to the excitation spectrum of the spinning string as studied in [68] (see also [69]).

There are many other questions left for the future. One is numerical solutions of the TBTBA equations, giving the interpolating function for the quark–antiquark potential and other interesting quantities in this model.

Another is to try to solve these equations at large coupling and reproduce semiclassical string theory results, as was done from the Y -system in [70]. Normally these techniques work only for large L and all other charges also large, since otherwise there is no semiclassical string description. In this case, though, the classical string description exists already for $L = 0$ as the usual description of Wilson loops in $AdS_5 \times S^5$. It should therefore be that the algebraic curves describing the string duals of these Wilson loops [13, 14, 15, 16, 6] can be derived from these equations.

The same tools used here can be used to study other open spin–chain models which arise in the AdS/CFT correspondence. The simplest are the D5-brane defects, with a very similar symmetry to the Wilson loop. There are two natural choices for the vacuum there, one with and one without boundary degrees of freedom. The latter should be essentially the same as the model studied here, while also the former should not be much different. One would need to calculate the boundary scalar factor and then always carry through the two boundary excitations, so similar to calculating excited states in this model.

Another interesting system to study is ABJM theory [71]. The 1/2 BPS Wilson loop is known in that theory [72], but the quantity calculated here — the quark–antiquark potential or cusp anomalous dimension has not been calculated in the gauge theory. The leading classical result from string theory on $AdS_4 \times CP^4$ is the same as for $AdS_5 \times S^5$ (with the appropriate identification between the string tension and the gauge coupling).

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A Supersymmetry

The vacuum of $\mathcal{N} = 4$ SYM is invariant under the $PSU(2, 2|4)$ superconformal group, summarized here following [38].

Denote by $\mathbf{L}^\alpha_\beta, \bar{\mathbf{L}}^{\dot{\alpha}}_{\dot{\beta}}$ the generators of the $SU(2)_L \times SU(2)_R$ Lorentz group, and by \mathbf{R}^A_B the 15 generators of the R -symmetry group $SU(4)$. The remaining bosonic generators are the translations $\mathbf{P}^{\dot{\beta}\alpha}$, the special conformal transformations $\mathbf{K}_{\alpha\dot{\beta}}$ and the dilatation \mathbf{D} . Finally the 32 fermionic generators are the Poincaré supersymmetries $\mathbf{Q}^\alpha_A, \bar{\mathbf{Q}}^{\dot{\alpha}A}$ and the superconformal supersymmetries $\mathbf{S}^A_\alpha, \bar{\mathbf{S}}_{\dot{\alpha}A}$.

The commutators of any generator \mathbf{G} with $\mathbf{L}^\alpha_\beta, \bar{\mathbf{L}}^{\dot{\alpha}}_{\dot{\beta}}$ and \mathbf{R}^A_B are canonically dictated by the index structure

$$\begin{aligned} [\mathbf{L}^\alpha_\beta, \mathbf{G}^\gamma] &= \delta^\gamma_\beta \mathbf{G}^\alpha - \frac{1}{2} \delta^\alpha_\beta \mathbf{G}^\gamma, & [\mathbf{L}^\alpha_\beta, \mathbf{G}_\gamma] &= -\delta^\alpha_\gamma \mathbf{G}_\beta + \frac{1}{2} \delta^\alpha_\beta \mathbf{G}_\gamma, \\ [\bar{\mathbf{L}}^{\dot{\alpha}}_{\dot{\beta}}, \mathbf{G}^{\dot{\gamma}}] &= \delta^{\dot{\gamma}}_{\dot{\beta}} \mathbf{G}^{\dot{\alpha}} - \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \mathbf{G}^{\dot{\gamma}}, & [\bar{\mathbf{L}}^{\dot{\alpha}}_{\dot{\beta}}, \mathbf{G}_{\dot{\gamma}}] &= -\delta^{\dot{\alpha}}_{\dot{\gamma}} \mathbf{G}_{\dot{\beta}} + \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \mathbf{G}_{\dot{\gamma}}, \\ [\mathbf{R}^A_B, \mathbf{G}^C] &= \delta^C_B \mathbf{G}^A - \frac{1}{4} \delta^A_B \mathbf{G}^C, & [\mathbf{R}^A_B, \mathbf{G}_C] &= -\delta^A_C \mathbf{G}_B + \frac{1}{4} \delta^A_B \mathbf{G}_C. \end{aligned} \quad (\text{A.1})$$

while commutators with the dilatation operator \mathbf{D} are given by $[\mathbf{D}, \mathbf{G}] = \dim(\mathbf{G}) \mathbf{G}$, where $\dim(\mathbf{G})$ is the dimension of the generator \mathbf{G} .

The remaining non-trivial commutators are

$$\begin{aligned} \{\mathbf{Q}^\alpha_A, \bar{\mathbf{Q}}^{\dot{\beta}B}\} &= \delta^B_A \mathbf{P}^{\dot{\beta}\alpha}, & \{\mathbf{S}^A_\alpha, \bar{\mathbf{S}}_{\dot{\beta}B}\} &= \delta^A_B \mathbf{K}_{\alpha\dot{\beta}}, \\ [\mathbf{K}_{\alpha\dot{\beta}}, \mathbf{Q}^\gamma_A] &= \delta^\gamma_\alpha \bar{\mathbf{S}}_{\dot{\beta}A}, & [\mathbf{K}_{\alpha\dot{\beta}}, \bar{\mathbf{Q}}^{\dot{\gamma}A}] &= \delta^{\dot{\gamma}}_{\dot{\beta}} \mathbf{S}^A_\alpha, \\ [\mathbf{P}^{\dot{\alpha}\beta}, \mathbf{S}^A_\gamma] &= -\delta^\beta_\gamma \bar{\mathbf{Q}}^{\dot{\alpha}A}, & [\mathbf{P}^{\dot{\alpha}\beta}, \bar{\mathbf{S}}_{\dot{\gamma}A}] &= -\delta^{\dot{\alpha}}_{\dot{\gamma}} \mathbf{Q}^\beta_A, \\ \{\mathbf{Q}^\alpha_A, \mathbf{S}^B_\beta\} &= \delta^B_A \mathbf{L}^\alpha_\beta + \delta^\alpha_\beta \mathbf{R}^B_A + \frac{1}{2} \delta^B_A \delta^\alpha_\beta \mathbf{D}, \\ \{\bar{\mathbf{Q}}^{\dot{\alpha}A}, \bar{\mathbf{S}}_{\dot{\beta}B}\} &= \delta^A_B \bar{\mathbf{L}}^{\dot{\alpha}}_{\dot{\beta}} - \delta^{\dot{\alpha}}_{\dot{\beta}} \mathbf{R}^A_B + \frac{1}{2} \delta^A_B \delta^{\dot{\alpha}}_{\dot{\beta}} \mathbf{D}, \\ [\mathbf{K}_{\alpha\dot{\beta}}, \mathbf{P}^{\dot{\gamma}\delta}] &= \delta^{\dot{\gamma}}_{\dot{\beta}} \mathbf{L}^\delta_\alpha + \delta^\delta_\alpha \bar{\mathbf{L}}^{\dot{\gamma}}_{\dot{\beta}} + \delta^\delta_\alpha \delta^{\dot{\gamma}}_{\dot{\beta}} \mathbf{D}. \end{aligned} \quad (\text{A.2})$$

To write down the integrable spin-chain for $\mathcal{N} = 4$ SYM one chooses a vacuum corresponding to one complex scalar field, usually labeled $Z = \Phi^5 + i\Phi^6$. This choice of spin-chain vacuum breaks the symmetry group $PSU(2, 2|4) \rightarrow PSU(2|2)^2$ with supercharges (Q^α_a, S^a_α) and $(\bar{Q}^{\dot{a}}_{\dot{b}}, \bar{S}^{\dot{a}}_{\dot{b}})$ where $a, b \in \{1, 2\}$ and $\dot{a}, \dot{b} \in \{3, 4\}$. Explicitly, the two copies of $\mathfrak{psu}(2|2)$ are the following subset of the generators of $\mathfrak{psu}(2, 2|4)$

$$\begin{aligned} Q^\alpha_a &= \mathbf{Q}^\alpha_a, & S^a_\alpha &= \mathbf{S}^a_\alpha, & L^\alpha_\beta &= \mathbf{L}^\alpha_\beta, & R^a_b &= \mathbf{R}^a_b - \frac{1}{2} \delta^a_b \mathbf{R}^c_c; \\ \bar{Q}^{\dot{a}}_{\dot{b}} &= \epsilon_{\dot{a}\dot{b}} \bar{\mathbf{Q}}^{\dot{\alpha}b}, & \bar{S}^{\dot{a}}_{\dot{b}} &= -\epsilon^{\dot{a}\dot{b}} \bar{\mathbf{S}}_{\dot{\alpha}b}, & \bar{L}^{\dot{\alpha}}_{\dot{\beta}} &= \bar{\mathbf{L}}^{\dot{\alpha}}_{\dot{\beta}}, & \bar{R}^{\dot{a}}_{\dot{b}} &= \mathbf{R}^{\dot{a}}_{\dot{b}} - \frac{1}{2} \delta^{\dot{a}}_{\dot{b}} \mathbf{R}^{\dot{c}}_{\dot{c}}. \end{aligned} \quad (\text{A.3})$$

In the presence of the Wilson loop or domain wall this is further broken down to a single copy of $PSU(2|2)$.

B The $\mathfrak{psu}(2|2)$ spin-chain

The spin-chain description of single trace local operators, which we will use also for the purposes of studying the boundary changing operators involves choosing a ground state,

usually taken to be $\text{Tr } Z^J$ with $Z = \Phi^5 + i\Phi^6$ and considering the excitations about it. The Choice of scalar Z breaks the symmetry group $PSU(2, 2|4) \rightarrow PSU(2|2)^2 \times U(1)$. Magnons are therefore classified by representations of the broken group. The basic magnons are in the fundamental representation of each of the $PSU(2|2)$, so we can first treat the magnons as if charged only under one of the groups, remembering to pair them up later.

When constructing magnon excitations on the spin-chain it is useful to consider the central extension of the $\mathfrak{psu}(2|2)$ algebras. The commutators of L^α_β , $\bar{L}^\alpha_{\dot{\beta}}$, R^a_b and $R^{\dot{a}}_{\dot{b}}$ are inherited from (A.1). The central extension for the Q and S commutation relations are

$$\begin{aligned} \{Q^\alpha_a, Q^\beta_b\} &= \epsilon^{\alpha\beta} \epsilon_{ab} P, & \{S^\alpha_a, S^\beta_b\} &= \epsilon_{\alpha\beta} \epsilon^{ab} K, \\ \{Q^\alpha_a, S^\beta_b\} &= \delta^\beta_a L^\alpha_\beta + \delta^\alpha_b R^b_a + \delta^\alpha_\beta \delta^b_a C, \end{aligned} \quad (\text{B.1})$$

and likewise for the second copy of $\mathfrak{psu}(2|2)$.

The fundamental representation has a pair of bosons ϕ^a and a pair of fermions ψ^α . The algebra acts on them by

$$\begin{aligned} Q^\alpha_a \phi^b &= a \delta^b_a \psi^\alpha, & Q^\alpha_a \psi^\beta &= -\frac{iaf}{x^-} \epsilon^{\alpha\beta} \epsilon_{ab} \phi^b, \\ S^\alpha_a \phi^b &= \frac{ia}{fx^+} \epsilon_{\alpha\beta} \epsilon^{ab} \psi^\beta, & S^\alpha_a \psi^\beta &= a \delta^\beta_a \phi^b, \end{aligned} \quad (\text{B.2})$$

where a , the spectral parameters x^\pm , the coupling g and the magnon momentum p are all related by

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}, \quad a^2 = ig(x^- - x^+), \quad e^{ip} = \frac{x^+}{x^-}. \quad (\text{B.3})$$

The central charges for this representation are given by

$$C = a^2 - \frac{1}{2}, \quad P = gf(1 - e^{ip}), \quad K = gf^{-1}(1 - e^{-ip}). \quad (\text{B.4})$$

For a single magnon we can eliminate the parameter f by rescaling $Q \rightarrow f^{1/2}Q$, $S \rightarrow f^{-1/2}S$ and $\psi \rightarrow f^{1/2}\psi$. It is crucial though for integrability when constructing multi-magnon states. In that case we take for the k^{th} magnon $f_k = e^{i \sum_{j=k+1}^M p_j}$. It is simple to show by induction that then for M magnons the total central charges are

$$P = \sum_{k=1}^M P_k = gf_M(1 - e^{i \sum_{k=1}^M p_k}), \quad K = \sum_{k=1}^M K_k = gf_M^{-1}(1 - e^{-i \sum_{k=1}^M p_k}). \quad (\text{B.5})$$

The S-matrix exchanges two fundamental representations and has the general form

$$\begin{aligned} s^{12} \phi_1^a \phi_2^b &= A^{12} \phi_2^{\{a} \phi_1^{b\}} + B^{12} \phi_2^{[a} \phi_1^{b]} + \frac{1}{2} C^{12} \epsilon^{ab} \epsilon_{\alpha\beta} \psi_2^\alpha \psi_1^\beta, \\ s^{12} \psi_1^\alpha \psi_2^\beta &= D^{12} \psi_2^{\{\alpha} \psi_1^{\beta\}} + E^{12} \psi_2^{[\alpha} \psi_1^{\beta]} + \frac{1}{2} F^{12} \epsilon^{\alpha\beta} \epsilon_{ab} \phi_2^a \phi_1^b, \\ s^{12} \phi_1^a \psi_2^\beta &= G^{12} \psi_2^\beta \phi_1^a + H^{12} \phi_2^a \psi_1^\beta, \\ s^{12} \psi_1^\alpha \phi_2^b &= K^{12} \psi_2^\alpha \phi_1^b + L^{12} \phi_2^b \psi_1^\alpha. \end{aligned} \quad (\text{B.6})$$

Up to an overall scalar phase, S_0 , all the terms in this matrix are fixed by symmetry [73, 74]. This is achieved by imposing that the left and right hand side transform in the same way under $\mathfrak{psu}(2|2)$. The solution is

$$\begin{aligned}
A^{12} &= S_0 \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}, & B^{12} &= A^{12} \left(1 - 2 \frac{1 - 1/x_2^- x_1^+}{1 - 1/x_2^+ x_1^-} \frac{x_2^- - x_1^-}{x_2^+ - x_1^-} \right), \\
D^{12} &= -S_0, & E^{12} &= D^{12} \left(1 - 2 \frac{1 - 1/x_2^+ x_1^-}{1 - 1/x_2^- x_1^+} \frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \right), \\
G^{12} &= S_0 \frac{x_2^+ - x_1^+}{x_2^- - x_1^+}, & C^{12} &= S_0 \frac{2a_1 a_2}{x_1^- x_2^-} \frac{f_2}{g} \frac{1}{1 - 1/x_2^+ x_1^-} \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}, \\
L^{12} &= S_0 \frac{x_2^- - x_1^-}{x_2^- - x_1^+}, & F^{12} &= S_0 \frac{2a_1 a_2}{x_1^+ x_2^+} \frac{1}{g f_2} \frac{1}{1 - 1/x_2^- x_1^+} \frac{x_2^+ - x_1^+}{x_2^- - x_1^+}, \\
H^{12} &= K^{12} = S_0 \frac{i a_1 a_2}{g} \frac{1}{x_2^- - x_1^+}.
\end{aligned} \tag{B.7}$$

C Reflection matrix

The reflection matrix is $\mathbb{R}(p) = R_0(p) \hat{\mathbb{S}}(p, -p)$, where \mathbb{S} is the scattering matrix written above with the factor of S_0 removed. $R_0(p) = \sigma_B(p)/\sigma(p, -p)$ is the boundary scalar factor discussed in Section 2.4 and evaluated in the next appendix. Under $p \rightarrow -p$ the spectral parameters transform as $x^\pm \rightarrow -x^\mp$. Therefore using the same ansatz as in (B.6), the coefficients of the reflection matrix are

$$\begin{aligned}
A_{12} &= R_0 \frac{x^-}{x^+}, & B^{12} &= -R_0 \frac{x^-((x^+)^3 + x^-)}{(x^+)^2(1 + x^+ x^-)}, \\
D^{12} &= -R_0, & E^{12} &= R_0 \frac{(x^+ + (x^-)^3)}{x^-(1 + x^+ x^-)}, \\
G^{12} = L^{12} &= R_0 \frac{x^+ + x^-}{2x^+}, & C^{12} &= i R_0 \frac{(x^+ + x^-)(x^+ - x^-)}{x^-(1 + x^+ x^-)}, \\
H^{12} = K^{12} &= -R_0 \frac{x^+ - x^-}{2x^+}, & F^{12} &= i R_0 \frac{x^-(x^+ + x^-)(x^+ - x^-)}{(x^+)^2(1 + x^+ x^-)}.
\end{aligned} \tag{C.1}$$

D Boundary dressing factor

D.1 derivation

The boundary dressing phase $\sigma_B(u)$ should satisfy the crossing and unitarity equations (2.21)

$$\sigma_B(u) \sigma_B(\bar{u}) = \frac{x^- + 1/x^-}{x^+ + 1/x^+}, \quad \sigma_B(u) \sigma_B(-u) = 1. \tag{D.1}$$

This equation can be solved the same way the bulk one is [40] (which is significantly simpler than the way it was originally found [75, 62]), see also [76]. Take the ansatz

$$\sigma_B(u) = e^{i(\chi_B(x^+) - \chi_B(x^-))}, \quad (\text{D.2})$$

and defining $\tilde{\sigma}_B(x) = e^{i(\chi_B(x) + \chi_B(1/x))}$, then using that under crossing transformations $x^\pm \rightarrow 1/x^\pm$ the crossing equation (D.1) can be written as

$$\sigma_B(u)\sigma_B(\bar{u}) = \tilde{\sigma}_B(x^+)\tilde{\sigma}_B(x^-) = \frac{x^- + 1/x^-}{x^+ + 1/x^+} = \frac{u - i/2}{u + i/2}. \quad (\text{D.3})$$

In terms of the shift operators $D^\pm = e^{\pm \frac{i}{2}\partial_u}$ this equation becomes

$$\tilde{\sigma}_B(x)^{D^+ - D^-} = (x + 1/x)^{D^- - D^+} = u^{D^- - D^+}. \quad (\text{D.4})$$

So

$$\tilde{\sigma}_B(x) = (x + 1/x)^{F[D]}, \quad F[D] \sim \frac{D^- - D^+}{D^+ - D^-} \quad (\text{D.5})$$

Taking $F[D] = -1$ will not lead to the desired answer, rather will introduce the false analytic structure. Instead one can try

$$F[D] = \frac{D^{-2}}{1 - D^{-2}} + \frac{D^{+2}}{1 - D^{+2}} = \sum_{n=1}^{\infty} (D^{-2n} + D^{+2n}) \quad (\text{D.6})$$

Therefore¹¹

$$\tilde{\sigma}_B(u) = \prod_{n=1}^{\infty} (u + in)(u - in) = \frac{\sinh \pi u}{\pi u} \quad (\text{D.7})$$

This expression does not quite work. The shifts by $\pm \frac{i}{2}$ gives

$$\tilde{\sigma}_B(u)^{D^+ - D^-} = \frac{\tilde{\sigma}(u + i/2)}{\tilde{\sigma}(u - i/2)} = \frac{\sinh \pi(u + i/2)}{\sinh \pi(u - i/2)} \frac{u - i/2}{u + i/2} = -(x + 1/x)^{D^- - D^+} \quad (\text{D.8})$$

This differs from (D.4) by a sign, which can be fixed by taking

$$\tilde{\sigma}_B(u) = \frac{\sinh 2\pi u}{2\pi u} \quad (\text{D.9})$$

This corresponds to the choice

$$F[D] = \frac{D^-}{1 - D^-} + \frac{D^+}{1 - D^+} = \sum_{n=1}^{\infty} (D^{-n} + D^{+n}) \quad (\text{D.10})$$

Of course one could replace the 2 in (D.9) with any even integer, but this would introduce extra poles.

¹¹The product is divergent, but this is the natural regularization of it.

Under crossing transformation $x \rightarrow 1/x$ so we can interpret this last equation as the discontinuity of χ_B across the cut in the u plane between $\pm 2g$. Therefore

$$\chi_B(u) = \int_{-2g+i0}^{2g+0i} \frac{dw}{2\pi i} \frac{x(u) - 1/x(u)}{x(w) - 1/x(w)} \frac{1}{u-w} \frac{1}{i} \log \frac{\sinh 2\pi w}{2\pi w}. \quad (\text{D.11})$$

Switching to the x coordinate and ignoring an irrelevant constant gives

$$\chi_B(x) = -i \oint \frac{dz}{2\pi i} \frac{1}{x-z} \log \frac{\sinh 2\pi g(z + 1/z)}{2\pi g(z + 1/z)}. \quad (\text{D.12})$$

The function $e^{-i\chi(x(u))}$ has a cut for $u \in [-2g, 2g]$ in the sheet where $|x(u)| > 1$. Crossing the cut gives $|x(u)| < 1$ where

$$e^{-i\chi(x(u))} = e^{i\chi(1/x(u))} \frac{2\pi u}{\sinh 2\pi u} \quad (\text{D.13})$$

On this sheet, in addition to the cut, this function has poles for all half integer imaginary u , except for $u = 0$.

D.2 Expansions

As is done with the bulk dressing factor, we can expand the boundary one for large x as¹²

$$i\chi_B(x) = \sum_{r=1}^{\infty} \frac{c_r(g)}{x^{2r}} \quad (\text{D.14})$$

Expanding (D.12) at large x are using $z = e^{i\psi}$ we find the integral expression

$$c_r(g) = \int_0^{2\pi} \frac{d\psi}{2\pi} e^{2ir\psi} \log \frac{\sinh 4\pi g \cos \psi}{4\pi g \cos \psi} \quad (\text{D.15})$$

This integral can also be written as

$$c_r(g) = 2(-1)^{r+1} \int_0^{\infty} \frac{J_{2r}(4gt)}{t(e^t - 1)} dt \quad (\text{D.16})$$

One can then expand at weak coupling

$$c_r(g) = \sum_{n=0}^{\infty} c_r^{(n)} g^{2n+2r} \quad (\text{D.17})$$

and the explicit factors are

$$c_r^{(n)} = -\frac{(-4)^{n+r}}{n+r} \frac{(2n+2r)!}{n!(n+2r)!} \zeta(2n+2r) \quad (\text{D.18})$$

¹²One can readily check that only even powers will appear in the expansion.

There is also an asymptotic strong coupling expansion

$$c_r(g) = \sum_{n=0}^{\infty} d_r^{(n)} g^{1-n} \quad (\text{D.19})$$

Apart for the linear and constant terms there are only odd inverse powers of g and the explicit factors are

$$\begin{aligned} d_r^{(0)} &= \frac{8(-1)^{r+1}}{4r^2 - 1} \\ d_r^{(1)} &= \frac{(-1)^r}{2r} \\ d_r^{(2n)} &= -\frac{16}{(4\pi)^{2n+1}} \Gamma(n + r - \tfrac{1}{2}) \Gamma(n - r - \tfrac{1}{2}) \zeta(2n) \quad n = 1, 2, \dots \\ d_r^{(2n+1)} &= 0 \quad n = 1, 2, \dots \end{aligned} \quad (\text{D.20})$$

One can now resum the series and express χ_B as

$$i\chi_B(x) = \sum_{n=0}^{\infty} d^{(n)}(x) g^{1-n} \quad (\text{D.21})$$

where

$$\begin{aligned} d^{(0)}(x) &= -4 + 4 \left(x + \frac{1}{x} \right) \operatorname{arccot} x \\ d^{(1)}(x) &= -\frac{1}{2} \log \left(1 + \frac{1}{x^2} \right) \\ d^{(2n)}(x) &= -\frac{16}{(4\pi)^{2n+1}} \Gamma(n + \tfrac{1}{2}) \Gamma(n - \tfrac{3}{2}) \zeta(2n) {}_2F_1(n + \tfrac{1}{2}, 1, \tfrac{5}{2} - n, -\tfrac{1}{x^2}) \quad n = 1, 2, \dots \end{aligned} \quad (\text{D.22})$$

The hypergeometric function is a rational function of x^2 .

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